Dirac Notation The Language of QM. (8)

- Since a vector is an object independent of the basis we use for its components, we want some notation that indicates the abstract object
  - In finite dimensions, we often use $\vec{x}$ but 1) we often think of that in terms of components and 2) want to encompass functions (...)
  - Write a time dependent state vector as a <u>ket</u> $|\Psi(t)\rangle$ or $|\Psi\rangle(t)$.
    We can also use a ket to represent the eigenstate of some operator, ignoring how it will evolve: $|x\rangle$, $|p\rangle$, $|E\rangle$, generally $|A\rangle$.
  - We can determine the components in any basis.
- <u>Dual vectors</u>:
  - In math, a dual vector is a linear function T that turns a vector into a scalar
  - Any such function is actually the inner product with some vector
    That is, there exists! $\vec{y}_T$ such that $T(\vec{x}) = \langle \vec{y}_T, \vec{x}\rangle$.
  - So dual vectors inter-relate with vectors.
    If $|\psi\rangle$ is some vector, define the <u>bra</u> $\langle\psi| \equiv (|\psi\rangle)^\dagger$ as the associated dual vector
  - Ex If $|\psi\rangle = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is a column vector, $\langle\psi| = [x_1^* \cdots x_n^*]$
    the row vector is the dual vector. This is the matrix adjoint
  - Since a dual vector acting on a vector is an inner product, we denote the inner product as $\langle\phi|\psi\rangle$.
    In our example $\langle\phi|\psi\rangle = [y_1^* \cdots y_n^*]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = y_1^* x_1 + \cdots + y_n^* x_n$
    In function Hilbert spaces, this is
    $$\langle\phi|\psi\rangle = \int d^n x\, \phi^*(x)\,\psi(x) \quad \text{over positions}$$

Normalization of states is just from this inner product.

- Sensible physical states satisfy $\langle\psi|\psi\rangle = 1$
- As we discussed, eigenstates of Hermitian operators can be made orthonormal basis
  - If the eigenvalues are discrete, this is easy $\langle\lambda_n|\lambda_m\rangle = \delta_{nm}$

    Energy in many cases is like this
  - It's a little more complicated for continuous spectrum of eigenvalues

    Consider momentum. Eigenfunctions $p\cdot\psi_p(x) = p\psi_p(x)$ are $\psi_p(x) = Ae^{ipx/\hbar}$ in 1D. The inner product is

$$|A|^2\int_{-\infty}^{\infty}dx\, e^{-ip'x/\hbar}\, e^{ipx/\hbar} = |A|^2 2\pi\hbar\, \delta(p'-p)$$

This tells us to normalize eigenstates $|p\rangle$, $|p'\rangle$ as

$$\langle p'|p\rangle = \delta(p'-p) \quad ; \quad \langle\vec{p}'|\vec{p}\rangle = \delta^3(\vec{p}'-\vec{p})$$

Similarly for position $\langle x'|x\rangle = \delta(x'-x)$

This is Delta-function normalization (or Dirac orthonormality)

- Note that these are not proper states: The closest thing to a position state, for ex, is a sharp wave function

  Delta-function normalized states act like an orthonormal basis, thoug

- Wave functions + states (Suppressing time dependence)
- If $|x\rangle$ makes a basis, we should be able to write any state $\langle x|x\rangle$ as a superposition $|\psi\rangle = \int dx'\,\psi(x')|x'\rangle$

  Therefore $\langle x|\psi\rangle = \int dx'\,\psi(x')\langle x|x'\rangle = \int dx'\,\psi(x')\delta(x-x') = \psi(x)$

  This is just how you get coordinates as inner products w/ basis vectors

  Worth repeating: $\psi(x) = \langle x|\psi\rangle$

- Similarly there is a momentum basis wave function

$$\psi(p) = \langle p|\psi\rangle \quad \text{the FT of } \psi(x)$$

- Or an energy-basis wave function $c_n = \langle E_n|\psi\rangle$
  Or any other observable, really
- Remember these are all just different representations of the same state

– Operators + the Completeness of Eigenstates

- Remember, an operator acting on a vector gives another vector in Hilbert space, but not necessarily a normalized one
  We therefore denote the action of an operator as $\mathcal{O}|\psi\rangle$
- Ex The Schrödinger equation is then $i\hbar \frac{d}{dt}|\Psi(t)\rangle = H|\Psi(t)\rangle$
  This is also just matrix multiplication in our examples
- This also gives us a simple notation for expectation values

$$\langle \mathcal{O}\rangle = \langle\psi|\mathcal{O}|\psi\rangle \leftarrow \text{check that this works}$$

But we can more generally sandwich an operator between any 2 states $\langle\phi|\mathcal{O}|\psi\rangle$ is some kind of generalized expectation
- And $\langle\phi|\mathcal{O}^\dagger|\psi\rangle = (\mathcal{O}|\phi\rangle)^\dagger|\psi\rangle$ by the definition of the adjoint
- Dyad operators: We can make an operator out of 2 states $\mathcal{O} = |\psi\rangle\langle\phi|$
  Ex You might hear of this as an "outer product" of vectors
  In matrix form, this is

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1^* & \cdots & y_n^* \end{bmatrix} = \begin{bmatrix} x_1 y_1^* & x_1 y_2^* & \cdots \\ \vdots & \ddots & \\ & & x_n y_n^* \end{bmatrix}$$

  + If $|e\rangle$ is a member of an orthonormal basis, $P = |e\rangle\langle e|$
  (or $P = \sum_i |e_i\rangle\langle e_i|$ where $\{|e_i\rangle\}$ is the subset of a basis)
  is a projection operator on to the appropriate subspace.
  Think about properties of this as an example.

- Now take **all** the members of a basis. This spans the whole Hilbert space, so $\sum_i |e_i\rangle\langle e_i| = \mathbb{1}$ (orthonormalized)

The generalization to delta-function normalization is clear

$$\mathbb{1} = \int dz\, |e_z\rangle\langle e_z|$$

Examples: we can clearly always insert the identity

$$|\psi\rangle = \int d^3\vec{x}\, |\vec{x}\rangle\langle\vec{x}|\psi\rangle \equiv \int d^3\vec{x}\, |\vec{x}\rangle\, \psi(\vec{x})$$

$$= \int d^3\vec{p}\, |\vec{p}\rangle\langle\vec{p}|\psi\rangle \equiv \int d^3\vec{p}\, |\vec{p}\rangle\, \tilde{\psi}(\vec{p})$$

$$= \sum_n |E_n\rangle\langle E_n|\psi\rangle = \sum_n c_n |E_n\rangle \quad \text{etc}$$

- This is a simple notation for a state in *any* basis
- Operators in finite-dimensional spaces are matrices

In bigger Hilbert spaces, we can think of matrix elements

$$\mathcal{O}_{ij} = \langle e_i|\mathcal{O}|e_j\rangle \quad \text{for orthonormal basis vectors } |e_i\rangle$$

By inserting dyad identity operators, we can turn everything into matrix multiplication

$$\mathcal{O}|\psi\rangle = \sum_{ij} |e_i\rangle \underbrace{\langle e_i|\mathcal{O}|e_j\rangle}_{\text{matrix}} \underbrace{\langle e_j|\psi\rangle}_{\text{vector of components}}$$

This was Heisenberg's matrix formulation of QM (as opposed to Schrödinger's wavefunction formulation)

People still call $\langle\phi|\mathcal{O}|\psi\rangle$ a matrix element

- Can you write out any observable $\mathcal{O}$ as a dyad of its eigenstates?

Hint: you should find $\mathcal{O} = \sum_n \lambda_n |\lambda_n\rangle\langle\lambda_n|$ for eigenvalues λ_n.

Can you now relate

$$\langle\mathcal{O}\rangle = \langle\psi|\mathcal{O}|\psi\rangle \text{ to } \langle\mathcal{O}\rangle = \sum_n \lambda_n P(\lambda_n)?$$