Relativistic Quantum Mechanics

- Rapid Relativity Review
  - Lorentz Transformations
    - Linear transformations with a special orthogonality property

    Include:
    - Boosts + Change in velocity of inertial reference frame
      + In x-direction
        $t' = \gamma(t - vx/c^2)$, $x' = \gamma(x - vt)$, $y' = y$, $z' = z$
    - Rotations + Or combinations thereof
      + Around z-axis $t' = t$, $x' = x\cos\theta + y\sin\theta$, $y' = -x\sin\theta + y\cos\theta$, $z' = z$
  - 4-vector Notation
    - It's convenient to write any Lorentz transformation as matrix multiplication + $X' = \Lambda X$
      + Use index notation $X^\mu = [ct, x, y, z]$, $\mu = 0, 1, 2, 3$

        $X^{\mu'} = \Lambda^{\mu'}{}_{\nu} X^\nu$ ~ Einstein summation convention = sum over paired "up" + "down" indices = contraction
    - An object that transforms this way is a (contravariant) 4-vector
      + Primary example X^μ + Energy-momentum vector $p^\mu = [E/c, \vec{p}]$
    - Lowering indices:
      + Define a metric tensor $\eta_{\mu\nu}$ $\equiv g_{\mu\nu}$ in flat spacetime. As a matrix $\eta = \mathrm{diag}(1, -1, -1, -1)$ (Griffiths conventions — I usually prefer the opposite)
      + $\eta_{\mu\nu}$ is invt under Lorentz transformations (by definition)
      + We can define a covariant 4-vector aka dual vector $a_\mu = \eta_{\mu\nu} a^\nu$
      + These transform in the opposite way as raised indices

        $$a_{\mu'} = (\Lambda^{-1})^T{}_{\mu'}{}^{\nu} a_\nu$$
      + You will see on HW that derivatives are an example.

— Scalar Product

- Fully contracted indices are Lorentz invariant

  + Ex $p \cdot x = p_\mu x^\mu = \eta_{\mu\nu} p^\nu x^\mu$

  + You can also raise indices with the inverse metric $\eta^{\mu\nu}$.
  Allows writing scalar product as $a \cdot b = a_\mu b_\nu \eta^{\mu\nu}$

- Includes physically important quantities

  + Momentum squared $p^2 = p \cdot p = (E/c)^2 - \vec{p}^{\,2} = (mc)^2 = (\text{rest mass})^2$

  $\Longrightarrow \quad E = \pm\sqrt{(pc)^2 + m^2c^4}$ (Note: you can have a sign)

  + phase of plane wave $e^{ip\cdot x/\hbar} = e^{-i(Et/\hbar)} e^{i\vec{p}\cdot\vec{x}/\hbar}$

- Relativistic Wave Equations

  — A caution: QM as we know it does not work in relativity very well.

  - You need rules for creating/destroying particles, which are usually packaged in Quantum Field Theory

  - The following wave equations should really be for operators (like the Heisenberg picture), but let's imagine them acting on wavefunctions.

  — Klein-Gordon Equation

  - The simplest way to make a relativistic equation would be to make a scalar product from partial derivatives

    + $\partial^2 \equiv \Box \equiv \eta^{\mu\nu} \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$

    + The K-G equation is

    $$(\partial^2 + \kappa^2)\Psi = 0$$

  - The solution is (for a single Fourier Mode)

    $$\Psi = A e^{-ik\cdot x} + B e^{ik\cdot x}, \quad k_\mu k^\mu = \kappa^2$$

+ In QM, a free particle wavefunction looks like

$$\Psi = e^{-iEt/\hbar} e^{i\vec{p}\cdot\vec{x}/\hbar} \Rightarrow k^\mu = [E/\hbar c, \vec{p}/\hbar] = P^\mu/\hbar$$

$$\Rightarrow \quad \mathcal{K} = mc/\hbar$$

- What's with the extra (negative frequency) solution?
  + This is extra b/c it's a 2nd-order DE. Doesn't match Schr. eqn.
  + More problematic: Probability $\int d^3\vec{x}\, |\Psi|^2$ is not conserved. Also b/c 2nd order

- Dirac Equation
- We want something that looks like $i\hbar \frac{\partial \Psi}{\partial t} = H\Psi$, so 1st order.

I'll write down an equation & argue it makes sense

  + We first want to postulate a set of matrices (Dirac Gamma Matrices) like Pauli spin matrices. Call them γ^μ and require

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$$

  + Like spin generates rotation transformations, these are related to Lorentz transformations of spin-1/2 particles.
  + There are many different forms (bases) for γ-matrices. Following Griffiths:

$$\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \gamma^{1,2,3} = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}$$ ← each entry is 2x2 block

- Now let Ψ be a 4-component vector called a Dirac spinor. This is not a 4-vector.

  + Consider $(i\gamma^\mu\partial_\mu - \mathcal{K})\Psi = 0$

  It's 1st order & can be written $i\hbar\frac{\partial\Psi}{\partial t} = \gamma^0(-i\hbar\vec{\gamma}\cdot\vec{\nabla} + mc)\Psi$ a lot like Schrödinger

+ It implies K-G eqn:

$$(i\gamma^\nu\partial_\nu + \kappa)(i\gamma^\mu\partial_\mu - \kappa)\Psi = (-\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu - \kappa^2)\Psi = 0.$$

Notice 1st term is

$$\tfrac{1}{2}(\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu + \gamma^\mu\gamma^\nu\partial_\mu\partial_\nu) = \tfrac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\partial_\mu\partial_\nu = \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial^2$$

$\Longrightarrow \ -(\partial^2 + \kappa^2)\Psi = 0$. So it satisfies $p^2 = (mc)^2$ ✓

- Solutions to the free Dirac Equation
  - Let's consider a Fourier Mode $\Psi = e^{-ip\cdot x/\hbar}\,u(p)$
    + Dirac eqn is $\gamma^\mu p_\mu u = mc\,u$
    + If $u = \begin{bmatrix} u_A \\ u_B \end{bmatrix}$ where $u_{A,B}$ = have 2 components,

$$u_A = \frac{c}{E - mc^2}(\vec{p}\cdot\vec{\sigma})\,u_B\,, \quad u_B = \frac{c}{E + mc^2}(\vec{p}\cdot\vec{\sigma})\,u_A$$

These are consistent for $E^2 = (pc)^2 + (mc^2)^2$.

+ Solutions

$$u = \begin{bmatrix} 1 \\ 0 \\ cp_z/E+mc^2 \\ c(p_x+ip_y)/E+mc^2 \end{bmatrix} \text{ or } u = \begin{bmatrix} 0 \\ 1 \\ c(p_x-ip_y)/E+mc^2 \\ -cp_z/E+mc^2 \end{bmatrix} \text{ for } E = +\sqrt{(pc)^2+(mc^2)^2}$$

or

$$u = \begin{bmatrix} cp_z/E-mc^2 \\ c(p_x+ip_y)/E-mc^2 \\ 1 \\ 0 \end{bmatrix} \text{ or } u = \begin{bmatrix} c(p_x-ip_y)/E-mc^2 \\ -cp_z/E-mc^2 \\ 0 \\ 1 \end{bmatrix} \text{ for } E = -\sqrt{(pc)^2+(mc^2)^2}$$

You can normalize these.

- Negative energy solutions?
  + Dirac says to imagine the ground state is full of "invisible particles" ← Dirac Sea
    You can add an electron on top (positive energy)
    or remove one, leaving a hole or anti-electron (positron) (negative energy)

    see reading
  + While the state has a negative energy, we say that the positron has a physical positive energy $p^\mu_{\text{physical}} = -p^\mu$