# Feynman Path Integral

- Review of Classical Mechanics (work in 1D)
  - Hamiltonian Formulation (familiar from Quantum)
    - Start with a Hamiltonian function of $x$ and $p$
      - Represents total energy $H(x,p) = K + V$
      - $p$ is the "canonically conjugate momentum" to $x$
    - Evolution determined by Hamilton's eqns

$$\dot{x} = \partial H/\partial p, \quad \dot{p} = -\partial H/\partial x$$

      - In our example with $K = p^2/2m$

$$\dot{x} = p/m, \quad \dot{p} = -\partial V/\partial x \equiv \text{Newton's Laws}$$

      - Related to Ehrenfest's theorem in QM
    - There is a known "quantization" procedure, though really one should derive classical from quantum.
  - Lagrangian Formulation
    - We have a Lagrangian $L(x, \dot{x}) = K - V$, $\dot{x} = dx/dt$
      - The integral of $L$ over time is the action functional

$$S[x(t)] = \int_{t_i}^{t_f} dt\, L(x(t), \dot{x}(t))$$

      - The action is a function of the full path $x(t)$. So you tell me how the particle moves as a function of time, I give you a single number = the action $S$.
    - Evolution determined by the "principle of least action"

$$\frac{\delta S}{\delta x(t)} = 0.$$

      - Formally, say $\dfrac{\delta x(t')}{\delta x(t)} = \delta(t'-t)$, $\dfrac{\delta \dot{x}(t')}{\delta x(t)} = \dfrac{d}{dt'}\delta(t'-t)$

Then do integrals

Example $L = \frac{1}{2}m\dot{x}^2 - V(x)$, $S = \int dt' \, L(x, \dot{x})$

$$\frac{\delta S}{\delta x(t)} = \int dt' \left( m\dot{x}(t') \frac{d}{dt'} \delta(t'-t) - \frac{dV}{dx}(x(t'))\, \delta(t'-t) \right)$$

$$= -m\ddot{x}(t) - \frac{dV}{dx}(x(t)) = 0 \quad \text{Makes sense!}$$

+ More informally, we can use $x + \delta x$ and write

$$S + \delta S = \int dt \, L(x+\delta x, \dot{x}+\delta\dot{x}) = \int dt \, L + \int dt \left( \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \right)$$

$$\Rightarrow \delta S = \int dt \left( \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right) \delta x \Rightarrow \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$

This is the same as above! Called Euler-Lagrange equations

+ Note: We've swept some stuff about boundary conditions under the rug.

- We can also define a canonical momentum + relate $L$ to $H$:
  + $p = \frac{\partial L}{\partial \dot{x}}$ much like $\dot{x} = \partial H / \partial p$
  + Then $H = L - p\dot{x}$ with $\dot{x}$ written in terms of $p$
    and $L = H - p\dot{x}$ with $p$ written in terms of $\dot{x}$
  + These two approaches are equivalent + equivalent to Newton's laws.

  A big advantage of Lagrangians is being able to change variables easily.
  Also manifests symmetries (like Lorentz invariance) clearly.

- QM in Lagrangian Form
  - Look at time evolution
    - $\langle \psi_f | e^{-iH(t_f - t_i)/\hbar} | \psi_i \rangle$ = amplitude to start in state $\psi_i$ at time $t_i$ and evolve to $|\psi_f\rangle$ at time $t_f$.

+ We can insert dyads

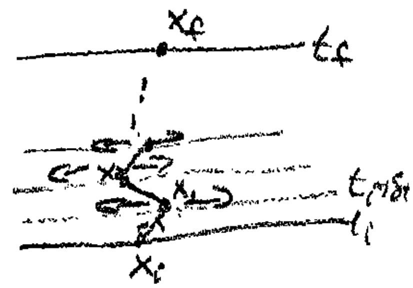
$$\int dx_f dx_i \langle \psi_f | x_f \rangle \langle x_f | e^{-iH(t_i - t_f)/\hbar} | x_i \rangle \langle x_i | \psi_i \rangle$$

where $|x_{i,f}\rangle$ are position eigenstates

+ We might as well just look at matrix elements

$$\mathcal{M} = \langle x_f | e^{-iH(t_f - t_i)/\hbar} | x_i \rangle$$

- Break up the interval $t_i$ to $t_f$ into $N$ steps of $\delta t$ each




+ Set $|x_f\rangle \equiv |x_{N+1}\rangle$, $|x_i\rangle \equiv |x_0\rangle$
+ Insert dyads $\int dx_n |x_n\rangle\langle x_n|$ at each timestep

$$\mathcal{M} = \int \prod_{n=1}^{N} dx_n \prod_{n=0}^{N} \langle x_{n+1} | e^{-iH\delta t/\hbar} | x_n \rangle$$

+ We start to see an "integral over all possible paths" emerging

- Now consider that $H = P^2/2m + V(x)$
  + We can write

$$e^{-iH\delta t/\hbar} = e^{-i(P^2/2m)\delta t/\hbar} e^{-iV(x)\delta t/\hbar} e^{\mathcal{O}(\delta t^2)} \leftarrow \text{commutator stuff}$$

+ As $\delta t \to 0$, we drop the $\mathcal{O}(\delta t^2)$ terms
+ Now insert a momentum dyad at each time step

$$\langle x_{n+1} | e^{-iH\delta t/\hbar} | x_n \rangle = \int dp_n \langle x_{n+1} | p_n \rangle \langle p_n | e^{-ip^2\delta t/2m\hbar} e^{-iV(x)\delta t/\hbar} | x_n \rangle$$

$$= \int dp_n \left( \frac{1}{\sqrt{2\pi}} e^{\frac{ip_n x_{n+1}}{\hbar}} \right) e^{-i(p_n^2/2m + V(x_n))\delta t/\hbar} \left( \frac{1}{\sqrt{2\pi}} e^{-ip_n x_n/\hbar} \right)$$

That replaces operators with eigenvalues and uses the momentum state wavefunction $\langle x|p\rangle$.

+ Putting it all together,

$$\mathcal{M} = \int \left(\prod_{n=1}^{N} dx_n\right)\left(\prod_{n=0}^{N} \frac{dp_n}{2\pi}\right) \exp\left[+i\frac{\delta t}{\hbar}\sum_{n=0}^{\infty}\left(p_n\dot{x}_n - \frac{p_n^2}{2m} - V(x_n)\right)\right]$$

where we used $p_n x_{n+1} - p_n x_{n} \approx p_n \dot{x}_n \delta t$.

+ We can call this a <u>phase-space</u> <u>path</u> <u>integral</u>

$$\mathcal{M} \equiv \int \mathcal{D}x\, \mathcal{D}p\; e^{i\int dt\,(p\dot{x} - H)/\hbar}$$

- We would really like just to integrate over $\mathcal{D}x$.
  + Notice that the $dp_i$ integrals are all Gaussian of the form

$$\int_{-\infty}^{\infty} dy\; e^{-by^2 + ay} = \sqrt{\frac{\pi}{b}}\, e^{a^2/4b} \quad \text{(by completing squares)}$$

  + Therefore,

$$\langle x_{n+1}| e^{-iH\delta t/\hbar} |x_n\rangle = \frac{1}{2\pi} e^{-iV(x_n)\delta t/\hbar} \int dp_n\, e^{-i\left(p_n^2/2m - p_n\dot{x}_n\right)\delta t/\hbar}$$

$$= \sqrt{\frac{m\hbar}{2\pi i\,\delta t}}\; e^{i\left(\frac{m}{2}\dot{x}_n^2 - V(x_n)\right)\delta t/\hbar}$$

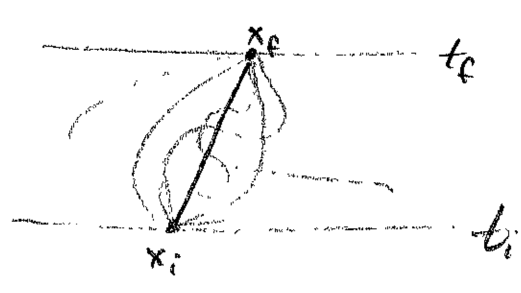
  + The whole thing is now

$$\mathcal{M} = \lim_{N\to\infty} \sqrt{\frac{m\hbar}{2\pi i\delta t}} \int \prod_{n=1}^{N}\left(\sqrt{\frac{m\hbar}{2\pi i\delta t}}\, dx_n\right) \exp\left[i\frac{\delta t}{\hbar}\sum_i\left(\frac{m}{2}\dot{x}_n^2 - V(x_n)\right)\right]$$

$$\equiv \int \mathcal{D}x\; e^{iS[x]/\hbar}$$

We'll ignore normalization concerns

- Heuristically, this is integrating over all possible paths between $x_i$ at $t_i$ and $x_f$ at $t_f$. <u>Feynman Path Integral</u> or <u>functional integral</u>



— Properties

- Really crazy mathematically: normalization is always divergent, but fortunately we can usually divide that out
- The variables in a path integral are <u>variables</u>, <u>not</u> <u>operators</u>
- Classical physics shows up cleanly
  - + If $S$ varies quickly, oscillating integrand cancels out
  - + Need $\delta S/\delta x(t) = 0$ to avoid that if $S/\hbar$ large

    That means classical physics dominates = saddle point approx.

- Uses of Path Integrals: You can calculate energy eigenvalues, but better for:

— Correlation Functions

- Haven't really talked about this, but we often want to look at expectation values of operators at different times
  - + Like $\langle x(t_2)\, x(t_1)\rangle$ ⟺ Need to keep track of time ordering

    $= \langle x e^{-iH(t_2-t_1)/\hbar} x \rangle$ etc.

  - + These tell you the response of one measurement to an earlier one (or one in a different location)
  - + Common in field theories, including material/condensed matter physics

    ex measure current at 2 different places

- These can be messy in Hamiltonian form. + As a path integral

$$\langle x(t_2)\, x(t_1)\rangle = \int \mathcal{D}x \; x(t_2)\, x(t_1)\, e^{iS[x]}$$

  - + There are nice tricks to simplify this integral, which really shine for harmonic oscillators + similar

— Semiclassical calculations of tunneling

Since it calculates a probability amplitude of going from $x_i$ to $x_f$, we can really look at decay rate in potentials like 

- Connection to Statistical Mechanics 

- In stat mech, we study the partition function $Z = \sum_n e^{-\beta E_n}$
  + In QM, write this as $Z = \sum_n \langle E_n | e^{-\beta H} | E_n \rangle$
  + If we insert position dyads

$$Z = \sum_n \int dx_i \int dx_f \, \langle E_n | x_f \rangle \langle x_f | e^{-\beta H} | x_i \rangle \langle x_i | E_n \rangle$$

  + But

$$\int dx_i \int dx_f \, \sum_n \langle x_i | E_n \rangle \langle E_n | x_f \rangle = \int dx_i \int dx_f \, \delta(x_i - x_f) = \int dx$$

so $$Z = \int dx \, \langle x | e^{-\beta H} | x \rangle$$

- That form looks like $\langle x_f | e^{-iHT/\hbar} | x_i \rangle = \int_{x(0)=x_i}^{x(T)=x_f} \mathcal{D}x \, e^{iS[x]/\hbar}$

except

1) $T = -i\beta\hbar$ 2) Paths are periodic $x(0) = x(\beta\hbar)$ w/ period $\beta\hbar$

  + Define $T = -i\beta\hbar$, $t \equiv -i\tau$; $\tau$ = "Euclidean time" b/c acts like space
  + Then $S = \int_0^T dt \left( \frac{1}{2} m \dot{x}^2 - V(x) \right) = -i \int_0^{\beta\hbar} d\tau \left( -\frac{1}{2} m \left( \frac{dx}{d\tau} \right)^2 - V(x) \right)$

$$= i \int_0^{\beta\hbar} d\tau \left( \frac{1}{2} m \left( \frac{dx}{d\tau} \right)^2 + V(x) \right) \equiv i S_E \leftarrow \text{Euclidean action}$$

  + The partition function is the Euclidean path integral

$$Z = \int_{\text{periodic}} \mathcal{D}x \, e^{-S_E/\hbar} \quad \text{over periodic paths}$$