# Formalism



## • Postulate 1 + Vector Spaces

States in QM are unit norm vectors in complex vector spaces with an inner product

→ What do we mean by a complex vector space? It is a set of vectors $\vec{x}, \vec{y}$;

1) closed under addition $\vec{x}+\vec{y}$ is a vector
   Addition is commutative and associative
2) There is a zero vector $\vec{O}$ s.t. $\vec{x}+\vec{O}=\vec{x}$
   and an inverse for each vector $\vec{x}+(-\vec{x})=\vec{O}$
3) closed under multiplication by complex scalars (numbers)
   $c\vec{x}$ is a vector
   Multiplication is distributive + associative
   Multiplication by $0$ gives $\vec{O}$ and by $1$ the vector $1\cdot\vec{x}=\vec{x}$.

Example A set of column matrices $\vec{x}=\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$

This has basis $\vec{e}_1=\begin{bmatrix}1\\0\\ \vdots\end{bmatrix}, \vec{e}_2=\begin{bmatrix}0\\ \vdots\\ \vdots\end{bmatrix}, \ldots$ ← smallest set of vectors whose linear combination spans the space

There may be many bases; the vector may be written in terms of any basis $\vec{x}=x_1\vec{e}_1+x_2\vec{e}_2+\ldots x_n\vec{e}_n$ or $\vec{x}=x_1'\vec{e}_1'+\ldots x_n'\vec{e}_n'$ ← components

Dimension = # of basis vectors

→ An inner product takes 2 vectors + gives a complex scalar $\langle\vec{x},\vec{y}\rangle$.

1) Linear in right-hand argument $\langle\vec{x}, a\vec{y}+b\vec{z}\rangle=a\langle\vec{x},\vec{y}\rangle+b\langle\vec{x},\vec{z}\rangle$
2) Anti-linear in left-hand argument $\langle a\vec{x}+b\vec{y},\vec{z}\rangle=a^*\langle\vec{x},\vec{z}\rangle+b^*\langle\vec{y},\vec{z}\rangle$
   Equivalently $\langle\vec{x},\vec{y}\rangle=\langle\vec{y},\vec{x}\rangle^*$
3) Positive Semi-definite $\langle\vec{x},\vec{x}\rangle\geq 0$ and equals only if $\vec{x}=\vec{O}$.

It allows us to define a norm $|\vec{x}|=\sqrt{\langle\vec{x},\vec{x}\rangle}$

Orthogonal vectors have $\langle\vec{x},\vec{y}\rangle=0$.

Example

~~For~~ n-dimensional example, $\langle\vec{x},\vec{y}\rangle=x_1^*y_1+x_2^*y_2+\ldots+x_n^*y_n$ in components if the basis is orthonormal. Just a dot product

Components in orthonormal basis are $x_i=\langle\vec{e}_i,\vec{x}\rangle$

Postulate 2 + Operators

Every observable is associated with a linear operator on the vector space of states
This operator is Hermitian, and any measurement gives an eigenvalue

— A linear operator $\mathcal{O}$ on the vector space:
 1) Takes a vector $\vec{x}$ to a vector $\vec{y}$ : $\vec{y} = \mathcal{O}\vec{x}$.
 2) Is linear $\mathcal{O}(a\vec{x} + b\vec{y}) = a\mathcal{O}\vec{x} + b\mathcal{O}\vec{y}$

- The (Hermitian) adjoint of $\mathcal{O}$ is denoted $\mathcal{O}^+$ and satisfies
 $\langle \vec{x}, \mathcal{O}\vec{y} \rangle = \langle \mathcal{O}^+\vec{x}, \vec{y} \rangle$. A Hermitean operator has $\mathcal{O}^+ = \mathcal{O}$.

Ex In any basis, $\mathcal{O}$ is represented by a matrix multiplying the column

$$\vec{y} = \mathcal{O}\vec{x} \implies y_i = \sum_j \mathcal{O}_{ij} x_j \quad \text{or} \quad y_i = \mathcal{O}_{ij} x_j$$

repeated index summed (Einstein convention)

Since $\langle \vec{x}, \vec{y} \rangle = x_i^* y_i$, $\langle \vec{x}, \mathcal{O}\vec{y} \rangle = x_i^* \mathcal{O}_{ij} y_j = (x_i \mathcal{O}_{ij})^* y_j$

This shows that $\mathcal{O}^+ = (\mathcal{O}^*)^T$

— An eigenvector or eigenstate of $\mathcal{O}$ is a non-zero vector $\vec{x}$ such that $\mathcal{O}\vec{x} = \lambda\vec{x}$ for $\lambda$ a scalar called the eigenvalue

Ex For a matrix representation, we have

$$(\mathcal{O} - \lambda I)\vec{x} = 0 \implies \det(\mathcal{O} - \lambda I) = 0$$

This equation gives the eigenvalues (+ the 1st then gives eigenvectors)

— Properties of Hermitian operators

- Eigenvalues are real: for eigenvector $\vec{x}$, of $\mathcal{O}$,

$$\langle \vec{x}, \mathcal{O}\vec{x} \rangle = \lambda \langle \vec{x}, \vec{x} \rangle \quad \text{and} \quad \langle \vec{x}, \mathcal{O}\vec{x} \rangle = \langle \mathcal{O}^+\vec{x}, \vec{x} \rangle = \langle \mathcal{O}\vec{x}, \vec{x} \rangle = \lambda^* \langle \vec{x}, \vec{x} \rangle$$

- Eigenvectors with different eigenvalues are orthogonal

$$\lambda_1 \langle \vec{x}, \vec{y} \rangle = \langle \vec{x}, \mathcal{O}\vec{y} \rangle = \langle \mathcal{O}\vec{x}, \vec{y} \rangle = \lambda_2 \langle \vec{x}, \vec{y} \rangle \implies \langle \vec{x}, \vec{y} \rangle = 0$$

- We will assume (and can sometimes prove) that eigenvectors of a Hermitian operator form a basis (which can be made orthonorma

## Postulate 3

If a system is in a state represented by a vector $\vec{X}$, then the probability of measuring the eigenvalue $\lambda$ of operator $\mathcal{O}$ with associated eigenvector $\vec{y}$ is $|\langle \vec{y}, \vec{x} \rangle|^2$

- This one is actually sort of self-explanatory. We will see how it relates to our old prescriptions shortly.

## Hilbert Spaces (remember this name)

- Vector spaces do not have to have finite dimensions

  We could imagine taking our column matrices + let them extend infinitely

- A Hilbert space is a vector space with an inner product that also includes the limit points of all series in it. • Basically, this extends the closure rule for addition of vectors to convergent infinite sums.
  - All the finite dimensional spaces we've talked about are Hilbert spaces
- The Hilbert spaces of interest in physics are called $L^2$ spaces over a given volume. These are spaces where the vectors are functions that are square-integrable (and therefore normalizable) over the volume

$$f(x) \in L^2 \text{ if } \int_V d^n x \, |f(x)|^2 < \infty$$

  - For example, this could be over all 3D volume or just an interval $a < x < b$ in 1D
  - The inner product is just $\langle f(x), g(x) \rangle = \int_V d^n x \, f^*(x) g(x)$

    And this is always finite for $L^2$ functions
- States + wavefunctions are normalized vectors in Hilbert space.