— Algebraic Representation

We have a finite # of states with any given l. Let's work these out as vectors + see algebraic relationships.

- First, let's reproduce the requirements on m, l
  - + There has to be a top state. That is, for $L_z|\lambda,\mu\rangle = \mu|\lambda,\mu\rangle$, $\vec{L}^2|\lambda,\mu\rangle = \lambda|\lambda,\mu\rangle$, $\mu^2 \leq \lambda$ b/c $\lambda = \langle L_x^2\rangle + \langle L_y^2\rangle + \mu^2 \geq \mu^2$
  - + That means, for the top state, $L_+|\lambda,\mu^+\rangle = 0$. And similarly for the bottom state $L_-|\lambda,\mu^-\rangle = 0$
  - + Since $L_\pm$ change μ by $\pm\hbar$, $\mu^+ - \mu^- = N\hbar$ for N some integer
  - + Finally, you can work out

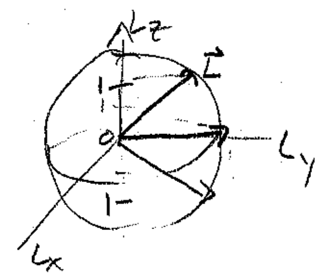
$$\vec{L}^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z$$

$$\Rightarrow \quad \lambda = \mu^+(\mu^+ + \hbar) = \mu^-(\mu^- - \hbar)$$

  - + This is solved for $\mu^- = -\mu^+$. But then $2\mu^+ = N\hbar$, so $\mu^+ = l\hbar$ where $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

$$\Rightarrow \lambda = l(l+1)\hbar^2 \text{ and a general } \mu = m\hbar, \quad m = \underbrace{-l, -l+1, \ldots, l-1, l}_{2l+1 \text{ states}}$$

  - + Two notes
    1) l, m can be $\frac{1}{2}$ integers (compare!)
    2) You cannot specify L_x, L_y simultaneously w/ L_z



- Next, some simple matrices. Work with fixed l
  - + For example $l=1$ has 3 states

$$|1,+1\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix} \quad |1,0\rangle = \begin{bmatrix}0\\1\\0\end{bmatrix} \quad |1,-1\rangle = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\vec{L}^2 = 2\hbar^2 \mathbb{1}, \quad L_z = \hbar\begin{bmatrix}1 & & 0\\ & 0 & \\ 0 & & -1\end{bmatrix}, \quad L_x, L_y \text{ on homework}$$

+ Don't forget $l=0$. One state $m=0$.

$$\vec{L}^2 = 0 \quad , \quad L_z = L_x = L_y = 0.$$

+ The process is similar for any total angular momentum l.
What about that funny $l = 1/2$? We can't write spherical harmonics, but we can write (2 states)

$$|1/2, +1/2\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad , \quad |\tfrac{1}{2}, -1/2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{L}^2 = \frac{3}{4}\hbar^2 \mathbb{1} \; ; \; L_z = \frac{\hbar}{2}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix} , \; L_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \; L_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Next, we have $L_+|\tfrac{1}{2},\tfrac{1}{2}\rangle = 0$, $L_+|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \hbar|\tfrac{1}{2},\tfrac{1}{2}\rangle \Rightarrow L_+ = \hbar\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Similarly $L_- = \hbar\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \Rightarrow L_x = \frac{L_+ + L_-}{2} = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $L_y = \frac{L_+ - L_-}{2i} = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

You can check that L_x, L_y have eigenvalues $\pm\hbar/2$ (like L_z)

+ Of course you can go on to $l = \frac{3}{2}, 2,$ etc.

- What's going on with the $l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$, etc states?
We know that these cannot be written as sensible wavefunctions of θ, ϕ.

+ Well, $\vec{L} = \vec{x} \times \vec{p}$ is angular momentum due to motion of the particle (say the electron). We call it <u>orbital</u> <u>angular momentum</u>.

+ Imagine that our electron is spinning. The spinning earth has angular momentum due to the orbital angular momentum of its parts.
The electron has no parts, but it can still have <u>intrinsic</u> <u>angular momentum</u> (spin). Reemphasize: spin is <u>built-in</u> to fundamental particles.
You can't stop them spinning.

+ Electrons, protons, neutrons, quarks all have ^total^ spin $1/2$.

+ This bit of strangeness ultimately has to do with how rotations are Lorentz transformations, so this is relativistic.

+ In string theory, spin does come from the motion of the string that makes the particle.

— The Physics of Spin.

- Just to clarify, we now have 2 sets of angular momentum ops:

$L_x, L_y, L_z, \vec{L}^2$ — name: orbital angular momentum; eigenvalues: ℓ = integer, m = integer

$S_x, S_y, S_z, \vec{S}^2$ — intrinsic or "spin"; $S^2 = \hbar^2 s(s+1)$, $S_z = m\hbar$ (sometimes m_s); s and m may be half-integer

  + These have the same commutators and same algebraic properties. Orbital angular momentum is restricted by the need to write sensible wavefunctions.
  + Spin is represented only by states or vectors/matrices. There is no way to write a sensible wavefunction.
  + spin is a new set of quantum numbers needed to describe states of particles with $s \neq 0$.

    For example: electron in 3D harmonic oscillator

    $|n_x, n_y, n_z\rangle |s=\tfrac{1}{2}, m\rangle$ or $|n, \ell, m\rangle |s=\tfrac{1}{2}, m_s\rangle$

    In a hydrogen atom, you have all that plus technically proton spin

- The spin ½ operators ← electrons have spin ½, same for protons
  + As we've said represent the states by 2-element vectors (sometimes called spinors)

$$|\tfrac{1}{2}, +\tfrac{1}{2}\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Any normalized state is

$$a|+\rangle + b|-\rangle = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{with } |a|^2 + |b|^2 = 1$$

  + As we worked out

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma} \quad \text{where } \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \ \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

σ_i are <u>Pauli spin matrices</u>

- Magnetic Fields + Spins

+ We've seen on homework how electromagnetic fields affect the Hamiltonian of a charged particle.

  But there is *another* very important effect:

  A spinning charged particle has a *magnetic dipole moment* $\vec{\mu} = \gamma \vec{S}$

  γ = *gyromagnetic ratio* $\simeq -e/m$ for electron

  From E+M, a dipole in magnetic field has Hamiltonian

$$H = -\vec{\mu} \cdot \vec{B} = -\gamma \vec{S} \cdot \vec{B}$$

+ An example: *Larmor precession*

  Put an electron at rest in $\vec{B} = B_0 \hat{z}$. $H = -\gamma B_0 S_z = -\frac{\gamma B_0 \hbar}{2} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$

  Eigenstates + energy eigenvalues

$$|+\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad E_+ = -\gamma B_0 \hbar/2, \quad |-\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad E_- = \gamma B_0 \hbar/2$$

  An initial state $|\Psi\rangle = \begin{bmatrix} \cos(\alpha/2) \\ \sin(\alpha/2) \end{bmatrix} \longrightarrow |\Psi(t)\rangle = \begin{bmatrix} \cos(\alpha/2)\, e^{i\gamma B_0 t/2} \\ \sin(\alpha/2)\, e^{-i\gamma B_0 t/2} \end{bmatrix}$

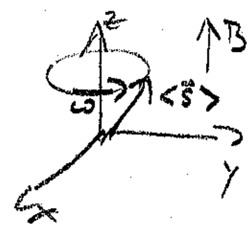
  As shown in the book, the expectation value is matrix multiplication

$$\langle S_z \rangle = \frac{\hbar}{2} \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \hbar/2 \left(|a|^2 - |b|^2 \right) \rightarrow \frac{\hbar}{2} \cos\alpha \quad \text{constant}$$

  But

$$\langle S_x \rangle = \frac{\hbar}{2} \sin\alpha \cos(\gamma B_0 t), \quad \langle S_y \rangle = -\frac{\hbar}{2} \sin\alpha \sin(\gamma B_0 t)$$

  So $\langle \vec{S} \rangle$ precesses at *Larmor frequency* $\omega = \gamma B_0$ around $\vec{B}$.

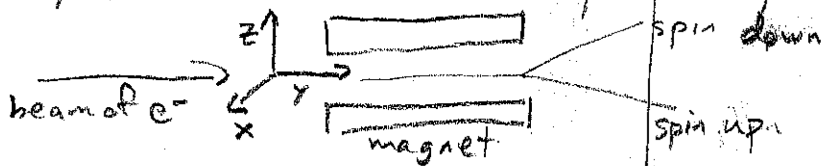


  Note: orientation b/c γ is negative

+ <u>Stern - Gerlach</u> <u>experiment</u> a famous historical example

If $\vec{B}$ is non-uniform, H depends on position $\Rightarrow$ force.

But sign/direction of force depends on spin. You can separate spins

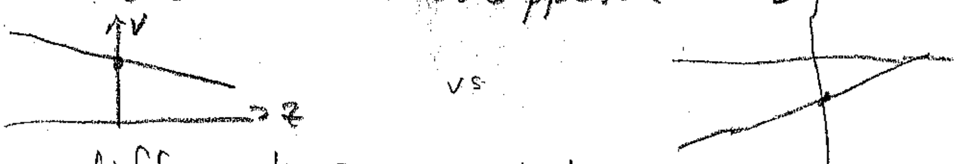


In magnet,

$$\vec{B} = (B_0 + \alpha z)\hat{z} - \alpha x \hat{x}$$

1) The x-component is needed for Maxwell's eqns but unimportant b/c Larmor precession averages it away for large B_0

2) Potential in z direction looks opposite for 2 spins



Generates different p_z as electron traverses magnet

3) Actual calculation a more complicated 3D scattering problem
Book gives heuristic calculation for a state $|\Psi\rangle = |\Psi_+\rangle|+\rangle + |\Psi_-\rangle|-\rangle$
Let's consider states $|+\rangle$ and $|-\rangle$ separately and look at $\langle p_z \rangle$.
By Ehrenfest,

$$\frac{d\langle p_z\rangle}{dt} = \frac{i}{\hbar}\langle [H, p_z]\rangle = \frac{i}{\hbar}(-\gamma\alpha)\langle \pm | S_z | \pm\rangle \langle \Psi_\pm | [z, p_z] | \Psi_+\rangle$$

$$= \pm \gamma \alpha \hbar / 2$$

4) So this is a method to measure spins of individual particles
Allows us to prove existence of spin 1/2 ← integer spins separate into odd # of spots on screen, half-integer into even.