

QM II Lecture Notes

(1)

Review Course Outline. NB <http://ion.uwinnipeg.ca/rafrey/qm2-11-12/>
Homeworks only on web page. Also make sure to check email (webmail)!
Explain how homeworks will work as well as participation.
Goals of class. Ask what they recall.

Review

① The Wave Function + Schrödinger's Equ.

- In classical mechanics, we solve everything deterministically, finding directly the motion of the particle
- Recall from QM I that microscopically particles are described not by phase space (position, momentum) but by a wavefunction
 $\Psi(\vec{x}, t)$ ← contains all information about the state
(The idea of state of a system also is fine in classical physics, but it is more abstract in QM)
- The wavefunction evolves according to Schrödinger's equation
$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V(\vec{x}) \Psi = H \Psi$$
 - This is a linear partial differential eqn, so we'll study these a lot this year
 - ∇^2 = Laplacian operator = $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ (etc in diff. dimensions)
 - H = Hamiltonian operator. As in class. mech., H describes the energy of the system. We will discuss operators more shortly.
- We will spend the first portion of the class renewing + amplifying your discussion of solutions to this equation from QM I

Physical Observables + Operators

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- The wave function is not something you can observe physically.
(In fact, it is generally a complex function)

- What are observables? Linear operators acting on the wavefunction.
(We'll certainly get back to this in more detail)

• As an example, define the expectation value of the position X (in 1D) as $\langle x \rangle = \int x |\Psi|^2 dx$ (Worry about what this means momentarily)

This is the operator $x: \Psi \rightarrow x\Psi$ in some kind of averaging procedure

• In analogy, define the expectation of any operator \hat{O} as $\langle \hat{O} \rangle = \int \Psi^* \hat{O} \Psi dx$

• Now consider momentum, which should satisfy $\langle \hat{p} \rangle = m \frac{d}{dt} \langle \hat{x} \rangle$

$$= m \int \hat{x} \left(\frac{\partial \Psi^*}{\partial t} \Psi + \Psi^* \frac{\partial \Psi}{\partial t} \right) dx = m \int \hat{x} \frac{i\hbar}{2m} \left(\Psi^* \nabla^2 \Psi - \nabla^2 \Psi^* \Psi \right) dx \text{ by Schr. eqn}$$

But we can rearrange to

$$= \frac{m i \hbar}{2m} \int \hat{x} \nabla \cdot (\Psi^* \nabla \Psi - \nabla \Psi^* \Psi) dx \xrightarrow{\text{int by parts twice}} = \frac{i \hbar}{m} \int \Psi^* \nabla \Psi dx$$

+ We identify the momentum operator $\hat{p} = -i\hbar \nabla$

+ Technically, we've been pretty sketchy with our boundary conditions

+ It seems that p and x are conjugate under Fourier transformation:

$$\text{FT} \left[\frac{d}{dx} \Psi \right] \approx \int \left(\frac{d}{dx} \Psi \right) e^{ikx} dx = ik \text{FT}[\Psi], \text{ etc. (ignoring normalization)}$$

$\hbar k = p$

• A general operator is built from position + momentum operators

EX

$$H = \frac{\vec{p}^2}{2m} + V(\vec{x})$$

- Any measurement yields an eigenvalue of the corresponding operator

That is, if you measure a value λ for operator \hat{O} , there is some eigenfunction

Ψ_λ such that $\hat{O} \Psi_\lambda = \lambda \Psi_\lambda$. That doesn't mean the wavefunction of the particle is that eigenfunction — at first.

Probabilistic Interpretation

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- Apparently it is an experimental feature of QM that we always get an eigenvalue of the operator as a result of measurement. But what lets us predict the value?
- Ψ assigns probabilities to each eigenvalue. Examples first
 - $|\Psi(x,t)|^2$ is probability density of measuring the position to be x .
That is $P(x_0, x_0 + \delta x) = \int_{x_0}^{x_0 + \delta x} |\Psi|^2 dx$ is the probability between x_0 and $x_0 + \delta x$.
 - Since we've seen that momentum is the Fourier transform of x , $|\text{FT}(\Psi)|^2$ is the probability density to measure a particular momentum. (integrated dp)
 - More generally, take $\int dx \Psi_2^* \Psi(x, \lambda)$. Then take $|_|_|^2$.
That's the probability (density) of measuring λ . More precise later.
- Sensible states must have total probability of 1.
 - Normalization $\int dx |\Psi(x,t)|^2 = 1$ independent of time
This is the only thing that sets a scale, since multiplying an eigenfunction by any complex number still gives you an eigenfunction of λ .
 - Schrödinger Eqn at least implies conservation of probability:
$$\frac{d}{dt} \int d^3x |\Psi|^2 = \frac{i\hbar}{2m} \int d^3x \vec{\nabla} \cdot (\Psi^* \vec{\nabla} \Psi - \vec{\nabla} \Psi^* \Psi) = 0$$
 by ^{usual} boundary conditions
- Before measurement, many values have non-zero probability. The measurement causes the wavefunction to collapse. Immediately after measuring a value λ , $\Psi(x) = \Psi_\lambda(x)$. How does this happen? Is this sensible? we'll discuss later, Expt. agrees.
↖ Copenhagen interpretation. Note: the particle has no position before measured. But a definite one afterward
- Probabilistic interpretation means that we imagine repeating the identical experiment an infinite # of times on an ensemble of identical systems. (not the same system over again)

- That tells us what we mean by probability above: the fraction of the time that we measure λ .
- But it also means that expectation values are averages over the ensemble of experiments

$$\langle \lambda \rangle = \sum \lambda P(\lambda)$$

You can see this directly for $\langle x \rangle = \int dx x (\text{Probability density})$

It is slightly more complicated to relate the 2 formulations (we'll see later how). For now, take $\langle O \rangle = \int d^3x \Psi^* O \Psi$ as a definition

- And it means that measurements have a spread. Consider $\Delta O = O - \langle O \rangle$. Clearly this averages to zero. But it has a variance

$$\sigma^2 = \langle \Delta O^2 \rangle = \dots = \langle O^2 \rangle - \langle O \rangle^2 \text{ (work it out!)}$$

σ is called uncertainty or standard deviation (think about Gaussians)

• Stationary States

- The Hamiltonian is an important operator bc it's in the Schrödinger eqn
- Its eigenvalues are energy $H \Psi_E(\vec{x}) = E \Psi_E(\vec{x})$ (since that's what it is classically)
- Its Eigenfunctions are stationary states. By the Schrödinger eqn

$$H \Psi_E = E \Psi_E = i\hbar \frac{\partial \Psi_E}{\partial t} \Rightarrow \Psi_E(\vec{x}, t) = \Psi_E(\vec{x}) e^{-iEt/\hbar}$$

- But a pure complex phase does nothing to probability densities or expectation values. All the physics of an E -eigenfunction is stationary!
- Only certain energies give normalizable solns to the time-independent Schrödinger Equation

$$\left(-\frac{\hbar^2}{2m} \nabla^2 + V(\vec{x}) \right) \Psi_E(\vec{x}) = E \Psi_E(\vec{x})$$

- Combinations of stationary states oscillate $\Psi = c_1 \Psi_1 e^{-iE_1 t/\hbar} + c_2 \Psi_2 e^{-iE_2 t/\hbar}$
work out probability density
- Much of QM is working out stationary states...