PHYS-4601 Homework 8 Due 10 Nov 2011

This homework is due in class on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

In this entire assignment, the physical system studied is the 1D harmonic oscillator.

It will be useful to remember the Gaussian integral

$$\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}} \ . \tag{1}$$

1. **Wavefunctions of the Harmonic Oscillator** *Mostly Griffiths 2.10*

   The text uses the condition that the ground state is annihilated by the lowering operator ($a|0\rangle = 0$) to find the ground state wavefunction $\langle x|0\rangle$. It also finds $\langle x|1\rangle$, the first excited wavefunction.

   (a) Apply the raising operator $a^\dagger$ to find the wavefunction $\langle x|2\rangle$ of the second excited state.

   (b) First show that the ground state and second excited state are orthogonal by using the ladder operators. Then carry out the same calculation by direct integration of the wavefunctions.

   (c) *Griffiths 2.31(a)* Consider a state with wavefunction

   $$\langle x|\psi\rangle = A\left(1 - 2\sqrt{\frac{m\omega}{\hbar}}x\right)^2 e^{-m\omega x^2/2\hbar} \tag{2}$$

   for A constant. What is the expectation value of the energy. *Hint:* You will need the wavefunctions of the first few states of the harmonic oscillator. Remember that there is a unique way to write any state as a superposition of energy eigenstates.

2. **Coherent States***Griffiths 3.35, roughly*

   In this problem, we will study *coherent states*, which are eigenfunctions of the lowering operator

   $$a|\alpha\rangle = \alpha|\alpha\rangle \ , \tag{3}$$

   where the eigenvalue α is generally complex.

   (a) Is there any energy eigenstate that is a coherent state? If so, list which energy eigenstate(s) are coherent and give the eigenvalue(s).

   (b) Find the expectation values of x and p in the coherent state $|\alpha\rangle$ (use the ladder operators).

   (c) Then find the uncertainties of x and p in $|\alpha\rangle$ and show that any coherent state is a state of minimal undertainty.

   (d) Show that a coherent state can be written as

   $$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle \ . \tag{4}$$

   To do that, you will first want to show that $\alpha\langle n|\alpha\rangle = \sqrt{n+1}\langle n+1|\alpha\rangle$. That gives you a recursion relation that the series (4) satisfies. Then you can check that $|\alpha\rangle$ is normalized (remember that $|n\rangle$ are orthonormal).

(e) Show that (4) is equivalent to

$$|\alpha\rangle = e^{-|\alpha|^2/2} \exp\left[\alpha a^\dagger\right] |0\rangle \ . \tag{5}$$

(f) If $|\alpha\rangle$ is the initial state of the system, show that the state at time t is still a coherent state with eigenvalue $\alpha(t) = \alpha e^{-i\omega t}$.