PHYS-4601 Homework 7 Due 27 Oct 2011

This homework is due in class on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **Gaussian Wavepacket Part II** *based on Griffiths 2.22*

   Here we return to the Gaussian wavepacket in 1D, here looking at the time evolution for the free particle Hamiltonian. We recall from the last assignment that the wavefunction (at some initial time) can be written as

   $$|\Psi(t=0)\rangle = \int_{-\infty}^{\infty} dx \left(\frac{2a}{\pi}\right)^{1/4} e^{-ax^2}|x\rangle = \int_{-\infty}^{\infty} dp \left(\frac{1}{2\pi a\hbar^2}\right)^{1/4} e^{-p^2/4a\hbar^2}|p\rangle \ . \tag{1}$$

   (a) Evolve this state in time. First write $\langle p|\Psi(t)\rangle$ and then show that

   $$\langle x|\Psi(t)\rangle = \frac{(2a/\pi)^{1/4}}{\sqrt{1+2i\hbar at/m}} e^{-ax^2/(1+2i\hbar at/m)} \ . \tag{2}$$

   Hint: You may want to use the trick of "completing the squares" to evaluate a Gaussian integral somewhere.

   (b) Find the probability density $|\langle x|\Psi(t)\rangle|^2$. Using the result from the last assignment that $\langle x^2\rangle = 1/4a$ at $t=0$, find $\langle x^2\rangle$ at a later time t by inspection of the probability density. Qualitatively explain what's happening to the wavefunction as time passes.

   (c) What's the momentum-space probability density $|\langle p|\Psi(t)\rangle|^2$? Does $\langle p^2\rangle$ change in time? Does this state continue to saturate the Heisenberg uncertainty relation for $t \neq 0$?

2. **Double Delta-Function Well** *based on Griffiths 2.27*

   Consider the potential

   $$V(x) = -\alpha\left[\delta(x+a) + \delta(x-a)\right] \ . \tag{3}$$

   As we did for the single delta-function well, define $\kappa = \sqrt{-2mE}/\hbar$.

   (a) Since $V(x)$ is an even function, a stationary state wavefunction is either even or odd. Find the even bound state wavefunctions ($\psi(-x) = \psi(x)$) and a transcendental equation for κ. Don't bother normalizing the wavefunction. Give a graphical argument to count the number of allowed energies for even bound state wavefunctions.

   (b) Find the odd bound state wavefunctions ($\psi(-x) = -\psi(x)$) and a transcendental equation for κ. What condition must α and a satisfy for an odd bound state to exist?

3. **The Probability Current and the Transmission Coefficient**

   Consider a conserved quantity Q (this could be electric charge or total probability in quantum mechanics) with a density ρ and current $\vec{\jmath}$. These quantities satisfy the *continuity equation*

   $$\frac{\partial\rho}{\partial t} = -\vec{\nabla}\cdot\vec{\jmath} \ . \tag{4}$$

   Remember that the divergence represents the lines of $\vec{\jmath}$ heading out through a surrounding surface (think of the divergence theorem and Gauss's law in electromagnetism). So this just says that that the change in the charge in a small volume is equal to the amount of charge flowing out through the surface surrounding the volume.

(a) Consider the probability density $\rho = |\Psi(x,t)|^2$. Use the Schrödinger equation to show that the probability current

$$\vec{\jmath}(x,t) = \frac{i\hbar}{2m}\left(\Psi\vec{\nabla}\Psi^* - \Psi^*\vec{\nabla}\Psi\right) \tag{5}$$

satisfies the continuity equation.

Suppose that we have a potential barrier or well in 1D with incoming wave $\Psi_{in} = Ae^{ikx-i\omega t}$, reflected wave $\Psi_{ref} = Be^{-ikx-i\omega t}$, and transmitted wave $\Psi_{trans} = Ce^{ik'x-i\omega t}$. The total wave-function to the left of the potential barrier/well is $\Psi_{in} + \Psi_{ref}$ and is just Ψ_{trans} to the right. Generally, if the potential is different on the different sides of the barrier/well, $k' \neq k$.

(b) Show that the probability current to the left of the barrier is $\vec{\jmath}_{in} + \vec{\jmath}_{ref}$ (the currents of Ψ_{in}, Ψ_{ref} respectively).

(c) Since the magnitude of the probability currents determine the probability carried off in the reflected and transmitted waves, we should define the reflection and transmission coefficients as

$$R = \left|\frac{\vec{\jmath}_{ref}}{\vec{\jmath}_{in}}\right| \;, \quad T = \left|\frac{\vec{\jmath}_{trans}}{\vec{\jmath}_{in}}\right| \;. \tag{6}$$

Find R and T in terms of the variables A, B, C, k, k'.