PHYS-4601 Homework 6 Due 20 Oct 2011

This homework is due in class on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **Heisenberg Equation**

   In the Schrödinger picture, states evolve in time according to the Schrödinger equation. In the Heisenberg picture, states are constant, but operators evolve in time. In this problem, you will find the differential equation that governs the evolution of an operator.

   Start by considering an operator with *no explicit time dependence*. In the Heisenberg picture, such an operator $\mathcal{O}$ satisfies

   $$\mathcal{O}(t) = e^{iHt/\hbar}\,\mathcal{O}(0)\,e^{-iHt/\hbar} \tag{1}$$

   for Hamiltonian H. Find the derivative $d\mathcal{O}/dt$ rigorously as follows:

   (a) First, by expanding the exponential as a power series and differentiating each term, find the time derivative of $\exp[\pm iHt/\hbar]$.

   (b) Then use the product rule to write $d\mathcal{O}/dt$ as a commutator. This is the Heisenberg equation (the equation $d\mathcal{O}/dt =$something).

   Two more short questions:

   (c) Is your Heisenberg equation consistent with Ehrenfest's theorem?

   (d) Consider the free particle Hamiltonian $H = p^2/2m$. Using the Heisenberg equation, find the operator $x(t)$ in terms of the operators $x(0)$ and $p(0)$.

2. **The Virial Theorem** *Based on Griffiths 3.31*

   Consider 3D quantum mechanics.

   (a) Using Ehrenfest's theorem, show that

   $$\frac{d}{dt}\langle\vec{x}\cdot\vec{p}\rangle = \left\langle\frac{\vec{p}^2}{m}\right\rangle - \left\langle\vec{x}\cdot\vec{\nabla}V(\vec{x})\right\rangle\ . \tag{2}$$

   (b) Show that the left-hand side of (2) vanishes in a stationary state to prove the *virial theorm*

   $$2\langle K\rangle = \left\langle\vec{x}\cdot\vec{\nabla}V(\vec{x})\right\rangle\ , \tag{3}$$

   where K is the kinetic energy. (The virial theorem holds classically, also, though without the expectation values.)

   (c) Using the virial theorem, find the ratio of (the expectation value of) the kinetic energy to the potential energy for the harmonic oscillator potential $V \propto \vec{x}^2$ and for the Coulomb potential $V \propto 1/|\vec{x}|$. *Hint:* Remember from electromagnetism (or Newton's law of gravity) that $\vec{\nabla}(1/|\vec{x}|) = -\vec{x}/|\vec{x}|^3$.

3. **Gaussian Wavepacket Part I**

   Here we take a first look at the Gaussian wavepacket in 1D, which is an important state in more than one physical system. In this problem, we will consider the state at a single instant $t = 0$, ignoring its time evolution. The state is

   $$|\psi\rangle = \int_{-\infty}^{\infty} dx\ Ae^{-ax^2}|x\rangle\ . \tag{4}$$

(a) Find the normalization constant A. *Hint:* To integrate a Gaussian, consider its square. When you square it, change the dummy integration variable to y, then change the integral over $dxdy$ to plane polar coordinates.

(b) Since the wavefunction is even, $\langle x \rangle = 0$. Find $\langle x^2 \rangle$. *Hint:* You can get a factor of x^2 next to the Guassian by differentiating it with respect to the parameter a.

(c) Write $|\psi\rangle$ in the momentum basis. *Hint:* If you have a quantity $ax^2 + bx$ somewhere, you may find it useful to write it as $a(x + b/2a)^2 - b^2/4a$ by completing the square. Then shift integration variables so it looks like you have a Gaussian again.

(d) Find $\langle p \rangle$ and $\langle p^2 \rangle$ and show that this state saturates the Heisenberg uncertainty principle.