PHYS-4601 Homework 5 Due 13 Oct 2011

This homework is due in class on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **Some Commutator Relations**

   For operators A, B, C:

   (a) show that
   $$[A, BC] = [A, B]C + B[A, C] \ . \tag{1}$$

   (b) prove the *Jacobi identity*
   $$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0 \ . \tag{2}$$

   (c) prove by induction that
   $$[A, B^n] = n[A, B]B^{n-1} \ , \tag{3}$$
   if $[A, B]$ commutes with B.

   Finally, consider the position and momentum operators, which have $[x, p] = i\hbar$.

   (d) Show using (3) that $[p, f(x)] = -i\hbar df/dx$.

2. **A Secret Look at Harmonic Oscillators**

   Consider some system (as we'll see in the weeks ahead, this could be a simple harmonic oscillator) where the orthonormalized energy eigenstates are labeled $|n\rangle$, where $n = 0, 1, 2, 3, \ldots$. Define the "lowering operator" a in terms of the dyad sum
   $$a = \sum_{n=1}^{\infty} \sqrt{n}|n-1\rangle\langle n| \ . \tag{4}$$

   (a) Find the Hermitian adjoint $a^\dagger$ of a and find the commutator $[a^\dagger, a]$. *Hint:* Remember how we do the adjoints of bras and kets. And also remember that Kronecker delta symbols allow you to eliminate summations.

   (b) Suppose the Hamiltonian can be written as $H = \hbar\omega a^\dagger a$. Find the energy eigenvalues.

   (c) Finally, define the position and momentum operators as
   $$x = \sqrt{\hbar/2m\omega}\left(a^\dagger + a\right) \ , \quad p = i\sqrt{\hbar m\omega/2}\left(a^\dagger - a\right) \ . \tag{5}$$
   Show that the ground state $|0\rangle$ gives the minimum possible uncertainty $\sigma_x\sigma_p = \hbar/2$.

3. **Incompatible Operators** *Griffiths 3.15*

   We call any two operators A and B incompatible if $[A, B] \neq 0$. Show that two incompatible operators cannot share a complete basis of eigenstates. *Hint:* Go about this by contradiction; show that if there exists a complete basis of states that are eigenstates of both A and B, then the action of $[A, B]$ on any state vanishes.