PHYS-4601 Homework 4 Due 6 Oct 2011

This homework is due in class on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

Ignore time dependence in this assignment.

1. **More Functions of Operators**

   (a) Suppose $|\lambda\rangle$ is an eigenfunction of $\mathcal{O}$, $\mathcal{O}|\lambda\rangle = \lambda|\lambda\rangle$. Show that

   $$f(\mathcal{O})|\lambda\rangle = f(\lambda)|\lambda\rangle \tag{1}$$

   for any function $f(x)$ that can be written as a power series.

   On the previous assignment, you showed that

   $$e^{-ipa/\hbar} \cdot \psi(x) = \psi(x-a) \tag{2}$$

   for any wavefunction $\psi(x)$. Please note that I have changed my conventions for the sign of a, but that doesn't change the relationship.

   (b) What are the eigenstates $|\lambda\rangle$ and eigenvalues of the translation operator $\exp[-ipa/\hbar]$?

   (c) Consider the wavefunction $\psi(x) = \langle x|\lambda\rangle$ for these eigenstates. Show directly that (2) is true for these states.

2. **Some Practice with Dirac Notation**

   We're just going to prove a few familiar facts in a couple of new ways. In the below, remember that position and momentum eigenstates have the inner product

   $$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \ . \tag{3}$$

   For simplicity, stick to one dimension.

   (a) Start with the wavefunction $\psi(x) = \langle x|\psi\rangle$ and insert a dyad $1 = \int dp\, |p\rangle\langle p|$ to show that ψ and the momentum space wavefunction $\tilde{\psi}(p) = \langle p|\psi\rangle$ are Fourier transforms of each other.

   (b) With the usual definition of the momentum operator, we know

   $$\langle x|p|\psi\rangle = -i\hbar \frac{d\psi}{dx} \tag{4}$$

   is the "wavefunction" of $p|\psi\rangle$. Show this by using the definition of the momentum operator as

   $$p = \int dp\, p|p\rangle\langle p| \ . \tag{5}$$

   (c) Using a similar argument, show that the action of the position operator x on the momentum space wavefunction $\tilde{\psi}(p)$ is $i\hbar d/dp$.

3. **Projectors and Dyad Operators**

   Start with a 3D Hilbert space with orthonormal basis $|1\rangle$, $|2\rangle$, $|3\rangle$. Consider 2 states

   $$|\alpha\rangle = i|1\rangle - 2|2\rangle - i|3\rangle \ , \quad |\beta\rangle = i|1\rangle + 2|3\rangle \ . \tag{6}$$

(a) Show that

$$|\alpha'\rangle = |\alpha\rangle - |\beta\rangle\langle\beta|\alpha\rangle/\langle\beta|\beta\rangle \tag{7}$$

is orthogonal to β. This process of orthogonalizing vectors is called the Gram-Schmidt process and can be used to construct an orthonormal basis given enough linearly independent vectors.

(b) *Griffiths problem 3.22(c)* Consider the operator $A = |\alpha\rangle\langle\beta|$. Is this Hermitean? Write A as a matrix in the orthonormal basis given.