PHYS-4601 Homework 3 Due 29 Sept 2011

This homework is due in class on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

Throughout this assignment, ignore time dependence of wavefunctions and suppress time in your notation. That is, don't bother writing out that a wavefunction or state depends on time.

1. **Boundary Conditions and Operators**

   Consider a particle in 1D confined to the line segment $0 < x < L$ (note that the Hamiltonian is not specified). All wavefunctions must satisfy Dirichlet boundary conditions $\psi(0) = \psi(L) = 0$. It is easy to see that functions with these boundary conditions and the usual inner product

   $$\langle\psi|\phi\rangle = \int_0^L dx\, \psi^*(x)\phi(x) \tag{1}$$

   form a Hilbert space.

   (a) Check that the momentum operator p satisfies the Hermiticity condition $\langle p\cdot\psi|\phi\rangle = \langle\psi|p\cdot\phi\rangle$.

   (b) Find the eigenstates and eigenvalues of the operator p^2 in this Hilbert space. Are any of the eigenstates of p^2 also eigenstates of the momentum operator p? Explain.

   (c) Presumably you have an apparent paradox. What is the resolution? *Hint:* Ask if p acting on any wavefunction in this Hilbert space always gives another wavefunction in this Hilbert space.

   (d) Now change the boundary conditions to *Neumann* boundary conditions ($d\psi/dx = 0$ at $x = 0, L$). Does p satisfy the Hermiticity condition? Is p a linear operator on this new Hilbert space?

   The lesson of this problem is to be careful with naive assumptions; boundary conditions can have a nontrivial effect.

2. **Probabilities and Densities**

   One of the postulates is that the probability (density) for measuring eigenvalue λ of some observable for a system in state $|\psi\rangle$ is $|\langle\lambda|\psi\rangle|^2$, where $|\lambda\rangle$ is the corresponding eigenstate of the observable. For the rest of this problem, consider a state $|\psi\rangle$. You may work in one dimension. *Hint:* You will find it useful to relate this abstract inner product to the usual one on wavefunctions.

   (a) Show that the probability density for measurements of position is the square of the absolute value of the wavefunction, as expected.

   (b) Show that the probability density to measure momentum p is given by the square of the absolute value of the Fourier transform of the wavefunction (with appropriate normalization).

   (c) Finally, find the probability of measuring energy E if the corresponding eigenstate has wavefunction $\psi_E(x)$.

3. **Unitary Operators**

   We've talked about Hermitian operators quite a bit. Unitary operators are another type of operator that are quite important in quantum mechanics. By definition, a unitary operator U satisfies $U^\dagger = U^{-1}$.

(a) First, show that if $U|\psi\rangle = \lambda|\psi\rangle$ (ie, $|\psi\rangle$ is an eigenstate of U), then $|\psi\rangle$ is an eigenstate of U^{-1} with eigenvalue $1/\lambda$ Then use this fact to show that an eigenvalue λ of a unitary operator U satisfies $|\lambda|^2 = 1$.

(b) Show that $U = \exp[iA]$ is unitary if the operator A is Hermitian. We define the exponential of an operator by a power series

$$\exp[iA] \equiv \sum_n \frac{1}{n!}(iA)^n = 1 + iA - \frac{1}{2}A^2 + \cdots \tag{2}$$

Hint: You may want to show that $(AB)^\dagger = B^\dagger A^\dagger$.

(c) Using the expansion above, argue that the operator $\exp[ipa/\hbar]$, where p is momentum and a is a constant, translates a wavefunction by a distance a. That is, show that

$$e^{ipa/\hbar} \cdot \psi(x) = \psi(x+a) \ . \tag{3}$$

So that exponential carries out translations. We will find that unitary operators often represent transformations like this. *Hint:* Think about the wavefunction's Taylor series around x.

4. **REMOVED**

5. **Homework Comments**

The following questions are **ungraded**, but your answers are greatly appreciated. This will be the last time I ask this, but you can always feel free to comment.

(a) On a scale of 1 to 10, with 1 being very easy, 10 very difficult, and 5 the average of homeworks from your physics classes last year, how difficult was this assignment?

(b) On a scale of 1 to 10, with 1 being very short, 10 very long, and 5 the average of homeworks from your physics classes last year, how long was this assignment?