PHYS-4601 Homework 21 Due 3 Apr 2012

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **The Path Integral and Ehrenfest's Theorem**

   As we've seen in the lecture notes, the probability amplitude of a transition from one given state at time t_i to another state at time t_f, as well as the expectation value of any variable $\mathcal{O}$ (called an operator in Hamiltonian quantum mechanics), can be given by the path integrals

   $$\int \mathcal{D}x\, e^{iS/\hbar}\ ,\quad \langle \mathcal{O}\rangle = \int \mathcal{D}x\, \mathcal{O}e^{iS/\hbar}\ , \tag{1}$$

   where S is the action of the system, given appropriate boundary conditions (which we intentionally leave vague). There is one other property of functional integrals that we haven't discussed yet:

   $$\int \mathcal{D}x\, \frac{\delta}{\delta x(t)}\,(\text{anything}) = 0\ , \tag{2}$$

   where $\delta/\delta x(t)$ is the functional derivative. This basically says that the integral of a total derivative is zero (we throw away boundary terms on configuration space; heuristically, we are integrating functions that go to zero as $x \to \pm\infty$).

   In this problem, consider a system with Lagrangian $L = (m/2)\dot{x}^2 - V(x)$ and action $S = \int dt\, L$.

   (a) Use (2) to show that the expectation value of x follows the classical Euler-Lagrange equation

   $$\left\langle \frac{\delta S}{\delta x(t)}\right\rangle = 0 \text{ or equivalently } m\frac{d^2}{dt^2}\langle x\rangle = -\left\langle \frac{dV}{dx}(x)\right\rangle\ . \tag{3}$$

   (b) Derive the Hamiltonian associated with the given Lagrangian. Then go back to our discussion of Ehrenfest's theorem from the fall (you can also find this in the Griffiths textbook). Show that the first order differential equations that determine $d\langle x\rangle/dt$ and $d\langle p\rangle/dt$ imply equation (3).

   (c) In this part, set the potential $V = 0$. By the method of your choice (*hint*: think about part (a)), show that the expectation value $\langle x(t)x(t')\rangle$ satisfies the differential equation

   $$\left(m\frac{d^2}{dt^2}\right)\langle x(t)x(t')\rangle = -i\hbar\delta(t-t')\ . \tag{4}$$

2. **Derivatives as 4-Vectors** *from Griffiths "Elementary Particles"*

   Suppose $f(x)$ is a scalar (Lorentz-invariant) function. Show that $\partial f/\partial x^\mu$ transforms as a covariant 4-vector. That is, if the Lorentz transformation of the coordinates is $x^\mu \to x'^\nu = \Lambda^\nu{}_\mu x^\mu$, then show that $\partial f/\partial x^\mu \to \partial f/\partial x'^\nu = (\Lambda^{-1})_\nu{}^\mu \partial f/\partial x^\mu$.

3. **Spin and Dirac Spinors** *from Griffiths "Elementary Particles"*

   With our choice of Dirac matrices, the spin operator is

   $$\vec{S} = \frac{\hbar}{2}\begin{bmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma}\end{bmatrix}\ . \tag{5}$$

   If the momentum $\vec{p}$ is along the z-axis, show that the spinor solutions u of the Dirac equation are S_z eigenvectors and find their eigenvalues.