PHYS-4601 Homework 2 Due 22 Sept 2011

This homework is due in class on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **Proofs About Stationary States**

   (a) *Rephrasing Griffiths 2.1(b)* Consider the spatial part of a stationary state $\psi(\vec{x})$ (that is, $\Psi(\vec{x},t) = \psi(\vec{x})e^{-iEt/\hbar}$). Show that $\psi(\vec{x})$ can be chosen real as follows. Argue that, for any ψ that solves the time-independent Schrödinger equation, so does ψ^*. Use that to show that the real and imaginary parts of ψ are also solutions with the same energy. Finally, argue that normalizability of ψ implies normalizability of its real and imaginary parts.

   (b) *Griffiths 2.2 rephrased* Suppose that the energy E of a stationary state in one dimension is less than the minimum value of the potential. Use the time-independent Schrödinger equation to show that the second derivative of the wavefunction always has the same sign as the wavefunction. Then use that fact to argue qualitatively that such a wavefunction cannot be normalized, proving by contradiction that E must be greater than the minimum value of the potential.

2. **Vector Space on a Circle**

   In this question, we consider a one-dimensional space where all functions must obey the periodicity condition $f(x) = f(x + 2\pi R)$. These are functions on a circle of radius R.

   (a) Prove that the (complex) functions on this circle form a vector space.

   (b) Prove that

   $$\langle f, g\rangle \equiv \int_0^{2\pi R} dx\, f^*(x)g(x) \tag{1}$$

   is an inner product.

   (c) Show that the free particle Hamiltonian $H = p^2/2m$ is Hermitian and find its eigenvectors (stationary states) and eigenvalues (energies). Do not prove that the stationary states form a basis for all functions but name the concept that shows they are a basis.

3. **A Two-State System**

   Consider some physical system which only has two states, so its states ψ can be represented by column vectors with two elements. In some basis, the Hamiltonian can be written as

   $$H = \hbar\omega \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \tag{2}$$

   (a) Find the stationary states and corresponding energies.

   (b) At time $t = 0$, the system is in state $\psi(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. What is the smallest positive value of t such that $\psi(t) \propto \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

   (c) Show that

   $$\psi(t) = \left(\cos(\omega t) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - i\sin(\omega t) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) \psi(0) \tag{3}$$

solves the time-dependent Schrödinger equation for an initial state $\psi(0)$. This is a specific case of a more general "time-evolution operator" we will talk about later.

4. **Homework Comments**

   The following questions are **ungraded**, but your answers are greatly appreciated.

   (a) On a scale of 1 to 10, with 1 being very easy, 10 very difficult, and 5 the average of homeworks from your physics classes last year, how difficult was this assignment?

   (b) On a scale of 1 to 10, with 1 being very short, 10 very long, and 5 the average of homeworks from your physics classes last year, how long was this assignment?