PHYS-4601 Homework 17 Due 8 Mar 2012

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **More Notes on Time-Dependent Perturbation Theory**

   In this problem, consider a Hamiltonian $H = H_0 + H_1(t)$, where we know the eigenstates $|\psi_n^0\rangle$ and eigenvalues E_n^0 of H_0 and where $H_1(t)$ is a small time-dependent contribution to the Hamiltonian. We write the full time-dependent state as

   $$|\Psi(t)\rangle = \sum_n c_n(t) e^{-iE_n^0 t/\hbar} |\psi_n^0\rangle \ . \tag{1}$$

   (a) In class, we derived the formula for $c_n(t)$ to first order in perturbation theory, assuming that $H_1(t)$ is small. Prove that those $c_n(t)$ satisfy the normalization condition $\sum_n |c_n|^2 = 1$ to first order in H_1 at all times. (See the lecture notes for the appropriate formula; equation [9.17] in Griffiths is *not* general enough for this problem.)

   (b) Suppose there are just two H_0 eigenstates $|1\rangle = |\psi_1^0\rangle$ and $|2\rangle = |\psi_2^0\rangle$ with perturbation Hamiltonian

   $$\langle 1|H_1|1\rangle = \langle 2|H_1|2\rangle = 0 \ , \quad \langle 1|H_1|2\rangle = \langle 2|H_1|1\rangle = \begin{cases} 0 & t < 0 \text{ or } t > T \\ V & 0 \le t \le T \end{cases} \ . \tag{2}$$

   If the initial state is $|\Psi(t=0)\rangle = |1\rangle$, find the probability that a measurement finds the system in state $|2\rangle$ at time $t = T$ to first order in H_1. *Hint*: You may directly do the integration or use the limit of the sinusoidal perturbation discussed in class.

2. **NMR aka MRI** *easier version of Griffiths 9.20*

   Consider a spin-1/2 particle (for example, a proton) with gyromagnetic ratio γ in the presence of a magnetic field

   $$\vec{B} = B_0 \hat{z} + B_1 \cos(\omega t)\hat{x} - B_1 \sin(\omega t)\hat{y} \ , \quad B_1 \ll B_0 \tag{3}$$

   at its fixed position. This is how an NMR (MRI) machine works; a proton sitting in a large static magnetic field is exposed to a small radio-frequency magnetic field. This problem is exactly solvable (as in the Griffiths problem), but you are to use first-order time-dependent perturbation theory.

   (a) The Hamiltonian is given by the usual interaction between a magnetic moment and magnetic field, $H = -\gamma \vec{B} \cdot \vec{S}$. Show that the Hamiltonian can be written as $H = H_0 + H_1(t)$, where H_0 has the same eigenstates as S_z with eigenvalues $\pm\gamma B_0 \hbar/2$ and

   $$H_1 = -\frac{\gamma B_1 \hbar}{2} \begin{bmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{bmatrix} \ . \tag{4}$$

   (b) At first order in the small field B_1, find the probability that a particle which is spin-up at $t = 0$ is measured to be spin-down at some later time t. For a given frequency ω, at what times is this probability maximized (that is, if we want to flip as many proton spins as possible, how long should we leave the radio-frequency pulse on)?

3. **Fermi's Golden Rule**

Consider a sinusoidal perturbation Hamiltonian $H_1 = Ve^{-i\omega t} + V^\dagger e^{+i\omega t}$. In the class notes, we found the probability for a transition from state $|1\rangle$ to $|2\rangle$ as a function of time and frequency ω. In the following, define $\hbar\omega_0 = E_2 - E_1$, the difference of the energy eigenvalues of the unperturbed Hamiltonian H_0. We will investigate the transition probability near $\omega = \omega_0$ at large t (at least as long as the probability stays small).

(a) At a fixed (and large) time, the probability is peaked at $\omega = \omega_0$. Using L'Hospital's rule or just a power series expansion, find the peak transition probability as a function of time.

(b) Find the values of ω where the probability first vanishes on either side of $\omega = \omega_0$. The difference in these two values tells us the width of the peak.

(c) For large enough times, approximate the transition probability as a rectangle with the peak value from part (a) and width given by half the difference in part (b). Integrate this approximate probability function and argue that

$$P \to \frac{2\pi|V_{21}|^2}{\hbar^2} t\delta(\omega_0 - \omega) \tag{5}$$

as $t \to \infty$.

This problem shows two things: first, transitions occur only to states at energies related by the perturbation frequency and, second, that there is a constant transition rate (probability per unit time) to the appropriate states. The relationship (5) is known as *Fermi's Golden Rule.*