PHYS-4601 Homework 17 Due 1 Mar 2012

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **Some More Details of Fine Structure**

   In this problem, we will fill in a few more details of the derivation of the first-order contribution to the hydrogen atom fine structure.

   (a) *Griffiths 6.12* Back in homework assignment 6, we showed that $\langle \vec{p}^2/2m\rangle = -(1/2)\langle V(\vec{x})\rangle$ for any stationary state of the Coulomb potential (virial theorem). Use the virial theorem to prove that
   $$\left\langle \frac{1}{r}\right\rangle = \frac{1}{n^2 a} \tag{1}$$
   (a is the Bohr radius as usual).

   (b) *Griffiths 6.17* Combine the first-order relativistic correction (lecture notes or Griffiths equation [6.57]) and the spin-orbit coupling correction (notes or Griffiths equation [6.65]) to the hydrogen atom stationary states in order to derive
   $$E_{n,j,m_j,\ell} = -\frac{mc^2\alpha^2}{2n^2}\left[1+\frac{\alpha^2}{n^2}\left(\frac{n}{j+1/2}-\frac{3}{4}\right)\right] , \tag{2}$$
   which is the total energy eigenvalue including the fine structure. Here, $\alpha = e^2/4\pi\epsilon_0\hbar c$ is the fine structure constant. *Hint:* It will help to write the (zeroth-order) Bohr energies in terms of the fine structure constant. You may also try to simplify using the fact that $j=\ell\pm 1/2$ and just working with both cases to get the same answer.

   (c) At the $n=2$ level, how many different energies are there, and what are their degeneracies?

2. **Weak-Field Zeeman Effect**

   In the class notes, we stated that placing a hydrogen atom in a constant magnetic field $B_0\hat{z}$ introduces a contribution to the hydrogen atom of $H_1 = (e/2m)B_0(L_z+2S_z)$. If this contribution is larger than the energy level splitting due to fine structure, this gives the "strong-field" Zeeman effect that we discussed in class. In this problem, consider the opposite limit, in which H_1 is smaller than the fine structure splitting. In this case, we include the fine structure corrections in the "unperturbed" Hamiltonian H_0 and treat H_1 as the perturbation to that.

   (a) With fine structure included, the eigenstates of H_0 are identified by n, total angular momentum quantum number j, its z component m_j, and the total orbital angular momentum quantum number ℓ (as well as total spin $s=1/2$); the z-components m_ℓ and m_s are not good quantum numbers. Write $H_1 = (e/2m)B_0(J_z+S_z)$ since $\vec{J}=\vec{L}+\vec{S}$ and show that the change in energy due to B_0 is
   $$E^1_{n,j,m_j,\ell} = \frac{e\hbar}{2m}B_0 m_j\left[1\pm\frac{1}{2\ell+1}\right] . \tag{3}$$
   To do this, you will need to know that the eigenstate of J^2, J_z, and L^2 is written
   $$\begin{aligned}|j=\ell\pm 1/2, m_j, \ell\rangle &= \sqrt{\frac{\ell\mp m_j+1/2}{2\ell+1}}|\ell, m_\ell = m_j+1/2, m_s=-1/2\rangle \\ &\pm\sqrt{\frac{\ell\pm m_j+1/2}{2\ell+1}}|\ell, m_\ell = m_j-1/2, m_s = 1/2\rangle \end{aligned} \tag{4}$$

in terms of the eigenstates of L^2, L_z, and S_z.

(b) The quantity in square brackets in (3) is called the Landé g factor. Show that the g factor can also be written as

$$\left[1 + \frac{j(j+1) - \ell(\ell+1) + 3/4}{2j(j+1)}\right] , \tag{5}$$

which is the form given in Griffiths. You can start with (5) and try $j = \ell \pm 1/2$ separately to get the form given in (3).

3. **Hyperfine Structure** *Griffiths 6.27*

Show that

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \sin\theta \hat{r}_i \hat{r}_j = \frac{4\pi}{3}\delta_{ij} , \tag{6}$$

where $\hat{r}_i$ is a component of the unit vector $\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$. Since the spherical harmonic $Y_0^0 = 1/\sqrt{4\pi}$ is constant, use this result to show that

$$\left\langle \frac{3(\vec{S}_p \cdot \hat{r})(\vec{S}_e \cdot \hat{r}) - \vec{S}_p \cdot \vec{S}_e}{r^3} \right\rangle = 0 \tag{7}$$

in $\ell = 0$ states of hydrogen (including the ground state). *Note:* You will not need to do any radial integrals in this problem!

4. **Anharmonic Oscillator**

Consider a particle moving in the potential

$$V(x) = \frac{1}{2}m\omega^2 x^2 + gx^3 , \tag{8}$$

where g is considered to be small, so this potential can be treated as a perturbation of a harmonic oscillator. Find the correction to the energy of the harmonic oscillator ground state $|0\rangle$ at both first and second order in g.