PHYS-4601 Homework 11 Due 5 Jan 2012

This homework is due in class on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.
There are no student presentations for this assignment.

1. **Pauli Spin Matrices** *Related to Griffiths 4.26*

   The Pauli spin matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} , \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1}$$

   are constructed to satisfy the commutator

$$[\sigma_i, \sigma_j] = 2i \sum_k \epsilon_{ijk} \sigma_k \ , \tag{2}$$

   where ϵ_{ijk} is the Levi-Civita symbol (as defined in class or under equation (4.153) in Griffiths). That gives the spin-1/2 operators $\vec{S} = (\hbar/2)\vec{\sigma}$ the correct commutators. This problem will explore another property of the Pauli matrices.

   (a) Show that the Pauli matrices also satisfy

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} \mathbf{1} \ , \tag{3}$$

   where δ_{ij} is the Kronecker delta symbol and $\mathbf{1}$ is the 2×2 identity matrix.

   (b) The four matrices $\mathbf{1}$ and $\sigma_{x,y,z}$ are linearly independent, so any 2×2 matrix A can be written as

$$A = a_0 \mathbf{1} + a_1 \sigma_x + a_2 \sigma_y + a_3 \sigma_z \tag{4}$$

   for some complex numbers $a_{0,1,2,3}$. Show that

$$a_0 = \frac{1}{2} \mathrm{Tr}(A) \ , \quad a_i = \frac{1}{2} \mathrm{Tr}(\sigma_i A) \quad (i = 1, 2, 3 = x, y, z) \ . \tag{5}$$

   Here, Tr is the *trace*, the sum of the diagonal elements of the matrix argument. *Hint*: The trace has the *cyclic property* $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for any two matrices A, B.

2. **Other Spin Components** *Almost Griffiths 4.29*

   In this question, work in the basis where the S_z eigenstates are $|+\rangle = [1, 0]^T$ and $|-\rangle = [0, 1]^T$.

   (a) Find the eigenvalues and eigenvectors of the operator $S_y = (\hbar/2)\sigma_y$ (see eqn (1) above).

   (b) If you measure S_y on a single particle, what possible values could you measure? What are the probabilities of those values in the general state $|\psi\rangle = a|+\rangle + b|-\rangle$ (a, b are complex numbers satisfying the normalization condition $|a|^2 + |b|^2 = 1$).

   (c) What is the expectation value of S_y in the state $|\psi\rangle$ of part (b)?

   Hint: Griffiths does some similar calculations in the text for the operator S_x.

3. **Rotations** *parts of Griffiths 4.56*

(a) Think back to our earlier problems on the translation operator. Argue that $\exp[i\varphi L_z/\hbar]$ is a rotation around the z axis by showing that

$$e^{i\varphi L_z/\hbar} \cdot \psi(\phi) = \psi(\phi + \varphi) \tag{6}$$

for any angular wavefunction $\psi(\phi)$ that can be written as a Taylor series around ϕ. *Hint*: This should be basically identical to what you did for the translation operator if you use the identification that $L_z = -i\hbar\partial/\partial\phi$.

As a result, the angular momentum operators are called the *generators* of rotations. In general, $\hat{n} \cdot \vec{L}/\hbar$ generates rotations around the unit vector $\hat{n}$. Furthermore, the rotations of spinors are generated by the spin angular momentum operators.

(b) What is the 2×2 matrix corresponding to a rotation of 2π around the z axis for spin 1/2? How does it compare to what you expected? *Hint*: In this and the next part, it pays to remember the result of equation (3) for products of the Pauli matrices.

(c) Construct the 2×2 matrix corresponding to a rotation of π around the x axis for spin 1/2. Show that it takes the S_z eigenstate $|+\rangle$ into $|-\rangle$.