PHYS-4601 Homework 10 Due 24 Nov 2011

This homework is due in class on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **Raising and Lowering**
   (a) *More or less Griffiths 4.18* Using the relation for $L_\pm L_\mp$ given in class and the text, show that
   $$L_\pm|\ell, m\rangle = \hbar\sqrt{(\ell \mp m)(\ell \pm m + 1)}|\ell, m \pm 1\rangle \ . \tag{1}$$
   (b) In a vector/matrix representation of the $\ell = 1$ states where
   $$|1,1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \ , \quad |1,0\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \ , \quad |1,-1\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \ , \tag{2}$$
   use (1) to find matrix representations of $L_\pm$ and then L_x and L_y.
   (c) *Griffiths 4.22(b)* Use $L_+ \cdot Y_\ell^\ell = 0$ and $L_z \cdot Y_\ell^\ell = \ell\hbar Y_\ell^\ell$ to determine $Y_\ell^\ell(\theta, \phi)$ up to overall normalization.

2. **Commutators and Things** *partly from Griffiths 4.19, partly inspired by problems in Ohanian*
   (a) Find the commutators $[L_z, x]$, $[L_z, y]$, $[L_z, p_x]$, and $[L_z, p_y]$.
   (b) Show that $\langle x\rangle = 0$ and $\langle p_x\rangle = 0$ for any eigenstate of L_z.
   (c) Show that $[L_z, H] = 0$ for Hamiltonian $H = \vec{p}^2/2m + V$ with central potential V. Argue that therefore $[\vec{L}^2, H] = 0$ also.
   (d) Find the uncertainty relation for the operators L_z and $\sin\phi$.

3. **Landau Levels** *from Griffiths 4.60*

   This problem considers the motion of electrons which are essentially confined to a 2D surface in the presence of an orthogonal magnetic field. This is the system used to describe the quantum Hall effect. Since this is a 2D problem, we won't include p_z (if you like, you can imagine that we consider only eigenstates of p_z with zero eigenvalue).
   (a) Show that a magnetic field $\vec{B} = B_0\hat{k}$ can be described by vector potential $\vec{A} = (B_0/2)(x\hat{\jmath} - y\hat{\imath})$. ($\hat{\imath}, \hat{\jmath}, \hat{k}$ are unit vectors along x, y, z respectively.)
   (b) We saw on the last assignment that the Hamiltonian is
   $$H = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 \ . \tag{3}$$
   In this case, show that we can write
   $$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{2}m\omega^2\left(x^2 + y^2\right) + \omega L_z \ , \tag{4}$$
   where $\omega = qB_0/2m$. Argue that $[H, L_z] = 0$ (you may use results from earlier problems).
   (c) Except for the L_z term at the end, this looks like a harmonic oscillator in the x and y directions. Write H and L_z in terms of the raising and lowering operators $a_x^\dagger, a_y^\dagger, a_x, a_y$ of those two harmonic oscillators. Evaluate $L_z|n_x, n_y\rangle$; is it diagonal?

(d) Apparently we have not yet found how to diagonalize H and L_z simultaneously. Now define lower operators

$$A = \frac{1}{\sqrt{2}}(a_y + ia_x) \ , \quad \bar{A} = \frac{1}{\sqrt{2}}(a_y - ia_x) \tag{5}$$

and their adjoints, the raising operators. First, show that $A, A^\dagger$ and $\bar{A}, \bar{A}^\dagger$ satisfy the usual commutation relations for raising and lowering operators. Then, find H and L_z in terms of $A, \bar{A}, A^\dagger, \bar{A}^\dagger$. From those expressions, argue that the energy eigenvalues are $E_n = \hbar\omega_B(n + 1/2)$, where $\omega_B = 2\omega = qB_0/m$ is the *cyclotron frequency*, and that the energy eigenstates are infinitely degenerate. These energy levels are called *Landau levels*; in practice, the finite size of the metal where the electrons live reduces the degeneracy to a finite amount.