Particle 4-Velocity

The idea is to write a 4-vector for velocity

- Notice that the proper time along the worldline of a particle is Lorentz invariant and spacetime position is a 4-vector.
  - Specifically, $d\tau = dt\sqrt{1-\vec{u}^2/c^2}$ is Lorentz invariant.
  - Also, $dx^\mu = (c\,dt, d\vec{x})$ is a 4-vector
  - Now consider $U^\mu = dx^\mu/d\tau$ derivative wrt. proper time

    The "denominator" is Lorentz invariant, "numerator" is 4-vector.

    Therefore $$U^{\mu'} = \frac{d}{d\tau}(\Lambda^{\mu'}{}_\nu x^\nu) = \Lambda^{\mu'}{}_\nu \frac{dx^\nu}{d\tau} = \Lambda^{\mu'}{}_\nu U^\nu$$
  - U^μ is a 4-vector. We call it 4-velocity.

- The components are interesting in terms of the normal velocity
  - $$U^0 = c\frac{dt}{d\tau} = \frac{c}{\sqrt{1-\vec{u}^2/c^2}} = c\gamma(\vec{u})$$
  - $$\vec{U} = \frac{d\vec{x}}{d\tau} = \frac{dt}{d\tau}\frac{d\vec{x}}{dt} = \gamma(\vec{u})\vec{u}$$

    Note that $|\vec{U}| \to \infty$ as $|\vec{u}| \to c$
  - Inverting this relationship, the coordinate velocity is
    $$\vec{u}/c = \vec{U}/U^0$$

- Comments + properties:
  - $U^0 > 0$ for any sensible particle traveling into the future
  - U^μ is always timelike and has square
    $$U^2 = U_\mu U^\mu = -c^2\gamma^2 + \gamma^2\vec{u}^2 = -c^2\gamma^2(1-\vec{u}^2/c^2) = -c^2$$

    This is true for <u>any</u> 4-velocity of <u>any</u> particle
  - This is really only defined for massive particles (ie, not photons)