4-Vectors

- The velocity transformation rule is too complicated. (And we didn't even look at something like acceleration.)
  - Why do we care? Remember, physical laws must be covariant:
    - The LHS of an equation must have the same transformation as the RHS.
    - In order to check that, we need well-organized sets of variables with simple transformation rules.
  - We've seen an analogous case: Rotations
    - Remember that vectors all have the same transformation rule under rotations. We wrote $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for coordinates and then $\vec{x}' = R\vec{x}$ for a rotation matrix R.
    - It is easy to check if equations are covariant because we know what quantities are vectors or scalars (objects that don't change under rotations)

      Vectors: position, velocity, acceleration, momentum, force ....

      Scalars: temperature, mass, energy, distance, ...
    - Then we can check if equations are covariant

      Newton: $\vec{F} = m\vec{a} \longrightarrow (R\vec{F}) = m(R\vec{a}) \Rightarrow \vec{F}' = m\vec{a}'$ ✓

      Coulomb: $\vec{F} = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}\hat{r} \longrightarrow (R\vec{F}) = \frac{q_1 q_2}{4\pi\varepsilon_0 (r^2)}(R\hat{r}) \Rightarrow \vec{F}' = \frac{q_1 q_2}{4\pi\varepsilon_0 (r')^2}\hat{r}'$ ✓

      We will explain more about the distance $r = r'$ again later

      Coulomb potential: $V(r) = q_1 q_2 / 4\pi\varepsilon_0 r$ ← all scalars, all invt.
  - Rotations are linear transformations on coordinates. So are the Lorentz boost transformations.

    We can make covariance of physics under Lorentz transformations easy if we use this analogy.

– We can turn our coordinates into "4-vectors" (Different notation from book)

$$\tilde{X} = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \quad \text{or} \quad x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3 = z$$

(like for $\vec{x}$, $x^1 = x, x^2 = y, x^3 = z$ but usually w/ subscripts)

- Lorentz boosts on 4-vectors:

  Write as matrix multiplication $\tilde{X}' = \Lambda \tilde{X}$

  \+ The matrix for a boost in standard configuration is

$$\Lambda = \begin{bmatrix} \gamma & -\gamma v/c & & \\ -\gamma v/c & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \quad \text{so} \quad \begin{aligned} ct' &= \gamma(ct) - (\gamma v/c)x \\ x' &= \gamma x - (\gamma v/c)(ct) = \gamma x - \gamma v t \end{aligned}$$

$y' = y$, $z' = z$ as usual.

  Work out the matrix multiplication

  \+ We can carry the analogy with rotations a bit farther.

$$\gamma^2 - (\gamma v/c)^2 = \frac{1}{(1 - v^2/c^2)}(1 - v^2/c^2) = 1 \quad \text{Like } \cosh^2\theta - \sinh^2\theta = 1$$

  Define $\cosh\theta = \gamma$, $\sinh\theta = \gamma v/c$. Then

$$\Lambda = \begin{bmatrix} \cosh\theta & -\sinh\theta & & \\ -\sinh\theta & \cosh\theta & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

  θ is called the <u>rapidity</u>

  \+ A 4-vector contains a normal vector $\tilde{x} = \begin{bmatrix} ct \\ \vec{x} \end{bmatrix}$.
  and rotations fit inside the bigger Lorentz transformation matrix

$$\Lambda = \begin{bmatrix} 1 & 0\ 0\ 0 \\ \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} & R \end{bmatrix} \ ; \quad \tilde{x}' = \Lambda \tilde{x} \implies \vec{x}' = R\vec{x},\ t' = t.$$

- Not every normal vector is a position.

  Anything that <u>rotates like position</u> <u>is a vector</u> (velocity, force, etc)

  Similarly, <u>anything that Lorentz boosts like spacetime position is a 4-vector</u>. For example, we will soon define a 4-vector velocity. Meanwhile, we can have 4-vectors $\tilde{a}$, $\tilde{b}$, etc.

– Scalar Products and Index Notation

There is a dot product that turns vectors into a scalar for rotations. We can do the same for Lorentz transformations

- For rotations, this 'scalar product' is the dot product

$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3 = \vec{a}^T\vec{b} \quad \text{la matrix multiplication}$$

+ Remember that this is invariant b/c rotation matrices have $R^T = R^{-1}$.

$$\vec{a}'\cdot\vec{b}' = \vec{a}'^T\vec{b}' = (R\vec{a})^T(R\vec{b}) = \vec{a}^T R^T R\vec{b} = \vec{a}^T(\mathbb{1})\vec{b} = \vec{a}\cdot\vec{b}$$

- We already have a invariant (scalar) that looks like a dot product: the invariant interval. Make a 4-vector $\delta\tilde{x}$

+ Then $\quad \delta s^2 = -c^2\delta t^2 + \delta\vec{x}^2 = -(\delta x^0)^2 + (\delta x^1)^2 + (\delta x^2)^2 + (\delta x^3)^2$

+ We can also write this as matrix multiplication.

Define

$$\eta = \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} \quad \text{so} \quad \delta s^2 = \delta\tilde{x}^T\eta\,\delta\tilde{x}$$

Sometimes call η the <u>metric</u>

+ Suppose you look at the invt. interval in another frame

$$(\delta\tilde{x}')^T\eta\,\delta\tilde{x}' = (\Lambda\delta\tilde{x})^T\eta(\Lambda\delta\tilde{x}) = \delta\tilde{x}^T(\Lambda^T\eta\Lambda)\delta\tilde{x}$$

This is equal δs^2 if and only if $\Lambda^T\eta\Lambda = \eta$. Check on HW

+ $\Lambda^T\eta\Lambda = \eta$ means that we can define a scalar product for any 2 4-vectors: $\tilde{a}\cdot\tilde{b} = \tilde{a}^T\eta\,\tilde{b}$

This is Lorentz invt: $\tilde{a}'\cdot\tilde{b}' = (\Lambda\tilde{a})^T\eta(\Lambda\tilde{b}) = \tilde{a}^T(\Lambda^T\eta\Lambda)\tilde{b} = \tilde{a}^T\eta\,\tilde{b}$

$$= (-a^0b^0 + a^1b^1 + a^2b^2 + a^3b^3)$$

+ Of course, that means you can square a 4-vector $\tilde{a}^2 = \tilde{a}\cdot\tilde{a}$.

We call

$\tilde{a}^2 < 0$ time like

$\tilde{a}^2 = 0$ light like

$\tilde{a}^2 > 0$ spacelike

Conventions often reversed.

- The property $\Lambda^T \eta \Lambda = \eta$ is similar to $R^T R = 1$
  + We called the rotations of 3D $O(3)$ for orthogonal matrices. When we dropped reflections ($\det R = -1$), we had $SO(3)$, special orthogonal matrices (special means $\det = +1$)
  + Since there are 1 time vs 3 space dimensions (that is, one entry -1 and 3 of $+1$ in η), we call the group of all matrices Λ that satisfy $\Lambda^T \eta \Lambda = \eta$ the Lorentz Group $O(3,1)$.
  + The Lorentz group contains rotations as well as boosts. In fact, two boosts in different directions "contains" a rotation. Repeat: boosts + rotations are tied together.
  + As with rotations, we can separate out reflections. We also want to remove time reversal $\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.
    So we take the special ($\det \Lambda = +1$) matrices that are orthochronous — the 0^{th} row, 0^{th} column element is positive
    We call this $SO^+(3,1)$. It is the "proper, orthochronous Lorentz group"
- But modern notation doesn't use matrix multiplication per se.
  + Matrix multiplication for rotations (for example) is

$$x^{i'} = \sum_j R^{i}{}_{j} x^j$$

← upper index is row #

← lower index is column #

The sum is adding across a row of R and down a column of x

Note that we put the prime on the index to represent the fact that the component is in the rotated frame

+ For 4-vectors, we will use Greek letters as indices $\mu, \nu = 0, 1, 2, 3$. Lower-case Roman i, j are always for space $i, j = 1, 2, 3$.

Then $\longrightarrow$ $$x^{\mu'} = \sum_\nu \Lambda^{\mu'}{}_\nu x^\nu$$ Again matrix multiplication

+ Einstein Summation Convention says leave off the summation symbol. If an index is repeated, you sum over it.

$\longrightarrow$ $$x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu$$ Call this index contraction

Note: we are now putting the prime on the index to represent frame S'. This is actually more appropriate since the vector is the same

+ Can you write a scalar product in index notation?

$$\tilde{a} \cdot \tilde{b} = a^\mu \eta_{\mu\nu} b^\nu = a^0 b^0 \eta_{00} + a^1 b^1 \eta_{11} + a^2 b^2 \eta_{22} + a^3 b^3 \eta_{33}$$
$$= -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$$

+ Why upper and lower indices?

We can define a new vector $a_\mu = \eta_{\mu\nu} a^\nu$

By components $a_0 = \eta_{00} a^0 = -a^0$, $a_i = \eta_{ij} a^j = a^i$.

How does this transform?

Well $$\tilde{a} \cdot \tilde{b} = a_\mu b^\mu = (b^\mu \eta_{\mu\nu} a^\nu) = a_{\nu'} b^{\nu'} = a_{\nu'} \Lambda^{\nu'}{}_\mu b^\mu$$

so $a_\mu = \Lambda^{\nu'}{}_\mu a_{\nu'}$. $a_{\nu'} = \bar{\Lambda}_{\nu'}{}^\mu a_\mu$ where $\bar{\Lambda}_{\nu'}{}^\mu \Lambda^{\rho'}{}_\mu = \delta^{\rho'}_{\nu'}$

As a matrix, this says $\bar{\Lambda} = (\Lambda^{-1})^T$.

+ So: Any upper index transforms by multiplying by $\Lambda^{\mu'}{}_\nu$ (contravariant)

Any lower index transforms by multiplying by $\bar{\Lambda}_{\mu'}{}^\nu$ (covariant)

If you contract a lower with an upper index, you get a Lorentz invariant quantity! (scalar)

+ Tensors are objects with multiple indices; each index transforms appropriately.

Example:
$$T_{\mu'\nu'}{}^{\lambda'} = \bar{\Lambda}_{\mu'}{}^{\alpha} \bar{\Lambda}_{\nu'}{}^{\beta} \Lambda^{\lambda'}{}_{\gamma} T_{\alpha\beta}{}^{\gamma}$$

The only tensor we know so far is the metric:
$$\eta_{\mu'\nu'} = \bar{\Lambda}_{\mu'}{}^{\alpha} \bar{\Lambda}_{\nu'}{}^{\beta} \eta_{\alpha\beta} \quad = \text{it's invariant}$$
(As a matrix eqn, this is $\eta = (\Lambda^T)^{-1} \eta \Lambda^{-1}$, equivalent to $\eta = \Lambda^T \eta \Lambda$)

Another example is the electromagnetic field.

- It is now easy to tell if an equation is allowed (meaning Lorentz covariant. Transformations are determined by indices. (if they are up or down).
  An equation is covariant if un-contracted indices are the same on all terms. Then it holds automatically in all frames.
  $$a^{\mu} f_{\mu\nu} g_{\gamma\lambda} = b_{\nu\lambda} \quad \checkmark \qquad u^{\alpha} u_{\beta} = p^{\mu} \quad \times$$
- From now on, we will write a 4-vector in terms of its components. So a^{μ} instead of $\tilde{a}$.