The Doppler Effect

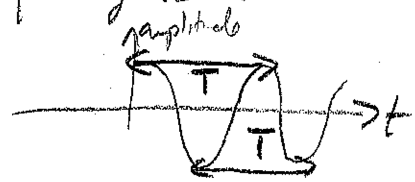
- The Galilean/Newtonian version for sound
  - Sound, of course, travels in a medium (say air) at speed c_s
    - Note: We will do a cursory treatment. Please see Barton for a more detailed version including ~~Galilean~~ relativity
    - As a consequence, the speed of the sound wave is relative to the rest frame of the wind.
      + So in a frame with wind velocity $\vec{w}$ along positive x,

$$c_s - w \longleftarrow (((\cdot))) \xrightarrow{\;w\;} \text{sound speed} = c_s + w$$

      + Of course, in the wind's frame

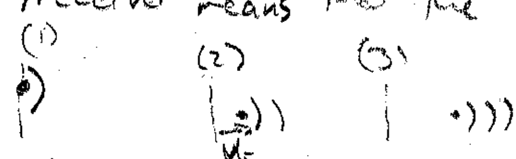
$$c_s \longleftarrow \underset{-w}{(((\cdot)))} \longrightarrow c_s$$

  - The collinear Doppler effect
    - The Doppler effect is the change of frequency of emitted sound vs the observed sound due to the motion of the emitter and receiver
      + ~~A~~ sound wave of definite frequency is a sine wave: regularly spaced peaks + troughs

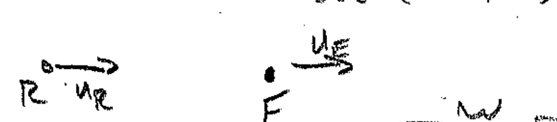


The period T is the time between peaks. Frequency $\nu = 1/T$. Angular frequency $\omega = 2\pi\nu$

      + The point is that motion of emitter/receiver means that the peaks catch up or spread out

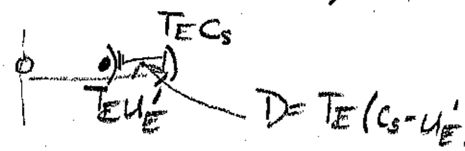


- ~~Calculate~~
  + ~~For~~ simplicity, we assume that wind velocity + velocities of emitter + receiver are all along x (positive or negative). Also take emitter to left of receiver to start



+ That's the "lab frame" S. Let's first work in the rest frame of the wind S'. Assume all $u_E', u_R', w \ll c_s$ (and $u_E, u_R \ll c_s$)

+ In the wind's rest frame, the emitter emits a peak and 1 period T_E later emits another. The distance between peaks is $D = T_E(c_s - u_E')$



+ Later, the 1st peak hits the receiver with the 2nd peak a distance D behind.

One "received period" T_R later, the 2nd peak hits.

But the 2nd peak must move $D + T_R u_R'$ to catch up. Therefore

$$c_s T_R = D + T_R u_R'$$

+ Eliminate D to find $\quad T_R(c_s - u_R') = T_E(c_s - u_E')$

Therefore, the ratio of frequencies is

$$\frac{\omega_R}{\omega_E} = \frac{T_E}{T_R} = \frac{c_s - u_R'}{c_s - u_E'}$$

At low speeds $u_E, u_R, w \ll c_s$, we can write

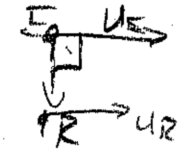
$$\frac{\delta\lambda}{\lambda} = \frac{u_E - u_R}{c_s} = \frac{|\vec{u}_E - \vec{u}_R|}{c_s}$$

+ To go to frame S, note that $u' = u - w$ for any object ; $\quad \dfrac{\omega_R}{\omega_E} = \dfrac{c_s - u_R + w}{c_s - u_E + w}$ (emitter to left)

+ There's a similar argument if emitter is to right of receiver

$$\frac{\omega_R}{\omega_E} = \frac{c_s + u_R - w}{c_s + u_E - w} \quad \text{(emitter to right)}$$

— Transverse Doppler effect



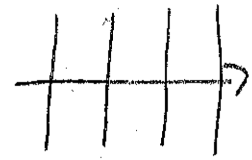
- Here we consider the wind velocity $w = 0$ and the relative velocity of E + R $\perp$ to separation.
- If separation distance is enough that the angle between separation + relative velocity remains perpendicular, the peaks/pulses don't get spread out or squashed. So $\omega_R / \omega_E = 1$ apparently.

- Relativistic Doppler Effect for Light

Could do a similar analysis based on emitter / receiver positions. But we have to work in extra effects (time dilation) anyway. Instead, we will use 4-vectors + relativistic dot products.

— Plane Waves in Relativity

- Plane waves are waves with planar wave fronts
  + ie, a peak or trough in amplitude fills a whole plane perpendicular to direction of travel



  + Can decompose into <u>modes</u> of a single frequency and <u>wavevector</u>

    Use angular frequency: $f_{\omega,\vec{k}}(t,\vec{x}) = \cos(\omega t - \vec{k}\cdot\vec{x} + \delta)$

    Can use sines or complex exponentials instead. This is <u>Fourier analysis</u>

  + Wave number $|\vec{k}| = 2\pi/\lambda$. Also tells direction of travel. Let $\vec{k} = k\hat{x}$

    If $\delta = 0$, there is a peak at $\omega t - kx = 0$. As $t\uparrow$, $x\uparrow$.

    Shows that the speed of the peak is $x/t = \omega/k$.

- Some important facts about light.
  + Does not need a medium, as we've said.
  + $\omega = |\vec{k}|c$ since the peaks move at the speed of light.

- Wave number 4-vector.
  + We can write a plane wave as $f_{\omega,\vec{k}} = \cos\left(\frac{\omega}{c}x^0 - \vec{k}\cdot\vec{x} + \delta\right)$

    Looks like $k\cdot x + \delta$ where $k^\mu = [\omega/c, \vec{k}]$.

  + Whether you are on a peak or a trough is a relativistic invt (does not depend on reference frame). So $k\cdot x$ is invt.

    That means k^μ is a 4-vector.

  + And k^μ is lightlike $k_\mu k^\mu = -(\omega/c)^2 + \vec{k}^2 = 0$.

— Doppler effect for light

- Set up: We have emitter E, receiver R, and light
  - 3 4-vectors are U_E^μ, U_R^μ, k^μ (4-velocities + wave 4-vector)
  - 2 reference frames: rest frame of R (S) + rest frame of E (S')

We want to find the received frequency $\omega_R = \omega$ in terms of $\omega_E = \omega'$

- We can make 6 relativistic scalars by taking the dot products
  - $k_\mu k^\mu = 0$, $U_{E\mu} U_E^\mu = U_{R\mu} U_R^\mu = -c^2$ tell us nothing new
  - $U_{R\mu} U_E^\mu$ tells us about the relative velocity but knows nothing about the light wave
  - So we might try $k_\mu U_E^\mu$ or $k_\mu U_R^\mu$.
- Let's try calculating the same product $k \cdot U_E = k_\mu U_E^\mu$ in both frames
  - In frame S', E is at rest $U_E^{\mu'} = [c, \vec{0}]$ and $k^{0'} = \omega'/c$.
    Therefore $k \cdot U_E = -\omega' \equiv -\omega_E$.
  - In frame S, E moves $U_E^\mu = [\gamma c, \gamma \vec{u}_E]$, $k^\mu = [\omega/c, \frac{\omega}{c}\hat{k}]$
    Therefore $k \cdot U_E = -\gamma\omega + \gamma \vec{u}_E \cdot \frac{\omega}{c}\hat{k} = -\gamma\omega(1 - \frac{\vec{u}_E}{c}\cdot\hat{k})$
  - Then the Doppler effect is
$$\frac{\omega_R}{\omega_E} = \frac{\omega}{\omega'} = \frac{1}{\gamma(1 - \vec{u}_E \cdot \hat{k}/c)} = \frac{\sqrt{1 - (\vec{u}_E/c)^2}}{1 - \hat{k}\cdot\vec{u}_E/c} \leftarrow \text{all in frame } S$$
  - Usually, we are at rest observing some light source, so it is convenient to use this frame S result.
    See text for result written in frame S' variables.

• Some Physics of the Doppler Effect

+ The square root factor is from time dilation. It's there as long as E + R have a nonzero relative velocity ($\vec{u}_E \neq 0$ in S)

+ At low relative velocities, we ~~often~~ write $\lambda_R = \lambda_E + \delta\lambda$, so

$$\frac{\lambda + \delta\lambda}{\lambda} = \frac{1 - \hat{k}\cdot\vec{u}_E/c}{\sqrt{1 - u_E^2/c^2}} \simeq 1 - \frac{\hat{k}\cdot\vec{u}_E}{c} \Rightarrow \frac{\delta\lambda}{\lambda} = -\frac{\hat{k}\cdot\vec{u}_E}{c}.$$

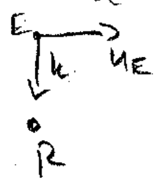
Much like sound.

Example: A star moves away from us.

⊕ $\overleftarrow{k}$ ✶ $\overrightarrow{u}$ For a single spectral line $\frac{\delta\lambda}{\lambda} = \frac{u_E}{c}$ $(\hat{k}\cdot\vec{u}_E = -u_E)$

This is a redshift, a change to longer wavelength

+ There is a transverse Doppler effect due to time dilation



$$\frac{\omega_R}{\omega_E} = \sqrt{1 - \left(\frac{u_E}{c}\right)^2} \quad b/c \quad \hat{k}\cdot\vec{u}_E = 0$$

+ A common effect is Doppler broadening

1) Suppose a star is rotating 
The light is blueshifted on 1 side + redshifted on the other.

2) You might have a bunch of objects emitting light but moving around (due to temperature, say) 
Astronomers can actually use this to measure the temperature of gas

+ A possible spectral line