QM I Lecture Notes

Course Outine information: NB http://ion.uwinnipeg.ca/~rafrey/qm1_11-12/
Homeworks only on web page Also make sure to check email! (webmail)
NB: Instructors will change (to M Martin) in Jan.

Goals of class

Special Relativity

Symmetries:

- A key concept in physics. - Name some: translations, time translations, rotations
  - Definition: Transformations that leave the laws of physics invariant (meaning, they look the same). Alternately, leaves the action the same
    - Ex · Newton's law $\vec{F} = m\vec{a}$. Rotationally invt b/c both sides vectors
      · Newton's law of gravity $\vec{F} = G\frac{MM}{r^2}\hat{r}$ Translationally invt b/c depends only on separation of particles
  - Take you from one frame of reference to another
- Why so important?
  - Each symmetry transformation leads to a conservation law! We will not prove this, but that is Nöther's theorem
    - Ex · Rotations $\longrightarrow$ Angular Momentum
      · Translations $\longrightarrow$ Momentum
      · Time Translation $\longrightarrow$ Energy
  - Can be used to build a theory of physics. That is, start with symmetry principle and ask what laws could possibly obey it.

★ We will use the Relativity Principle
- All laws of nature have the same form in all inertial frames

or

- No expt performed in one inertial frame can determine its motion relative to another inertial frame.

- The math of symmetry transformations
  - There's always an identity transformation, "do nothing"
  - There is always an inverse transformation to a given transformation
    Meaning: you can go back and forth between reference frames
  - There is a composition or "multiplication" rule
    which is closed: two symmetry transformations performed in sequence give another one
  - Symmetry Composition is associative in the same sense as multiplication
  - This means symmetries make up a group ← branch of mathematics

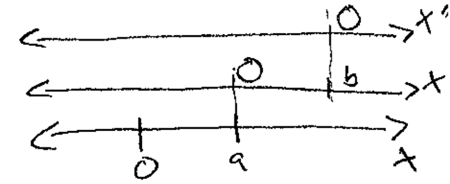
Ex
- Translations are real numbers $\mathbb{R}$ with addition as composition
- 2D rotations are reals w/ addition mod 2π.
  or complex numbers of unit modulus with multiplication aka $U(1)$
- 3D rotations: given by matrices acting on 3D vectors $\vec{x} \Rightarrow R\vec{x}$. $\vec{x}' =$
  These are orthogonal (preserve dot products) $O(3)$ (or $SO(3)$)
  $$\vec{y}\cdot\vec{x} = \vec{y}'\cdot\vec{x}' \Rightarrow [-y-]\begin{bmatrix} \vert \\ x \\ \vert \end{bmatrix} = [-y'-]\begin{bmatrix} \vert \\ x' \\ \vert \end{bmatrix} = [-y-]\, R^T R \begin{bmatrix} \vert \\ x \\ \vert \end{bmatrix}$$
  - We will discuss these in a little more detail later.

- An aside: we've only mentioned space-time symmetries.
  Can you think of a different type of conservation law?
  Electric charge ← the symmetry comes from the QM of electrons
  this is from the ability to set electrostatic potential to 0 at any p[oint]
  and it leads to the electromagnetic fields + force

- Active vs Passive Transformations:
  - We might think of looking at the same process from 2 frames, for example, choosing your time origin appropriately ← These are passive
  - Or you could think of a single frame but 2 related processes
    ie. particle moving along x or along y ← active
  - Little difference, usually used interchangeably.

Additional Notes on Symmetry Group Examples

- Translations - Let's just look at 1 dimension



Start with coordinates x and shift the origin of x' by a distance a to the right

Then for any point, $x' = x - a$. The inverse is from shifting back to x.
$x = x' + a$

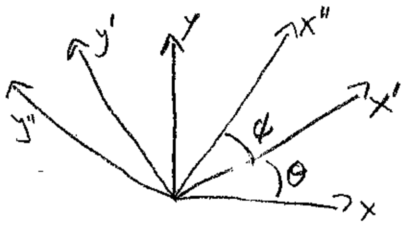
The composition rule comes from shifting to x'' b to the right of x'

$$x'' = x' - b = (x-a) - b = x - (a+b)$$ Composition is addition

Since you can translate by any real number, the group is $(\mathbb{R}, +)$ (real numbers with addition)

Figuring out the identity element is left as an exercise.

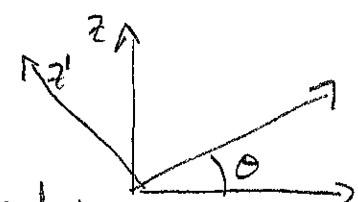
- 2D rotations (rotations around a fixed axis)



As you can see from the figure, rotations are composed by adding the angles (a rotation by θ then ϕ is a rotation by $\theta + \phi$)

But $\theta \cong \theta + 2\pi$ ($\cong$ means "equivalent to") so the group is $(\mathbb{R}, + \bmod 2\pi)$

Alternately, write the (x,y) plane as a complex plane for $z = x + iy$



Notice that the argument of all z along the real z' axis is θ but the argument of z' along that axis is 0

Therefore $z' = z e^{-i\theta}$. Rotations are multiplication by a phase

In this form, composition is complex multiplication of phases

We call this group $U(1)$.

The 1 means 1×1 matrices (or complex numbers)

The U means "unitary": matrices such that the Hermitian adjoint $U^\dagger = (U^T)^*$ is the inverse. $(e^{-i\theta})^\dagger = e^{i\theta} = (e^{-i\theta})^{-1}$

- **3D rotations**

We can write coordinates in 3D as vectors $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Since a rotation mixes components in a linear way, a rotation is matrix multiplication acting on a vector $\vec{x}' = R\vec{x}$.

For example the rotation matrix for rotation around z is

$$R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ (it leaves the z-coordinate alone)

Composition is matrix multiplication. That is, rotation by R_1 and then R_2 is rotation $R_3 = R_2 R_1$.

Rotation matrices are real and also preserve dot products

$$\vec{x}' \cdot \vec{y}' = \vec{x} \cdot \vec{y}.$$

In matrix notation, $\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y}$ (ie, row vector matrix multiplying column vector — work this out if you are not familiar with this)

That means

$$\vec{x}' \cdot \vec{y}' = (\vec{x}')^T (\vec{y}') = (\vec{x}^T R^T)(R\vec{y}) = \vec{x}^T \vec{y}$$

because the dot product is preserved. Therefore $R^T R = 1$

In words, the matrix transpose is the inverse rotation.

These matrices are called orthogonal so the 3D rotation group is called $O(3)$ — 3×3 orthogonal matrices.

See the first homework assignment for more about these.