PHYS-3301 Homework 8 Due 9 Nov 2011

This homework is due in class on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

A Major Reminder: These questions are both to teach you things but also to get you practice with the index notation and the Einstein summation convention. For that reason, I am using both exclusively.

1. **Raising an Index**

   In the class notes, we studied 4-vectors with superscript indices a^μ and defined 4-vectors with subscript indices through the lowering process
   $$a_\mu = \eta_{\mu\nu} a^\nu \ . \tag{1}$$
   (a) Invert the relationship (1). In other words, write $a^\mu = \beta^{\mu\nu} a_\nu$ and find the matrix $\beta^{\mu\nu}$.

   (b) Show that
   $$\eta_{\mu\lambda}\eta_{\nu\rho}\beta^{\lambda\rho} = \eta_{\mu\nu} \ . \tag{2}$$
   Hint: It is may be easiest to start by showing that $\eta_{\mu\lambda}\beta^{\lambda\rho} = \delta^\rho_\mu$, the Kronecker delta symbol. In either case, you want to remember that $\eta_{\mu\nu}$ is nonzero only when $\mu = \nu$.

   As a result of part (b), we see that β and η are the same tensor, just with indices in different positions. From now on, we will call $\beta^{\mu\nu}$ by $\eta^{\mu\nu}$ instead. One major point of this problem is to show you that raised and lowered indices are interchangeable; they are just related by $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$.

   (c) Calculate $\eta^{\mu\nu}\eta_{\mu\nu}$. You should get a pure number.

2. **Lowered Indices**

   As discussed in the class notes, 4-vectors with raised or lowered indices have the following Lorentz transformations:
   $$a^{\mu'} = \Lambda^{\mu'}{}_\nu a^\nu \ \text{ and } \ a_{\mu'} = \bar{\Lambda}_{\mu'}{}^\nu a_\nu \ , \tag{3}$$
   where Λ is the usual Lorentz transformation matrix and $\bar{\Lambda}^T = \Lambda^{-1}$ as a matrix.

   (a) Show that the matrix relationship between $\bar{\Lambda}$ and Λ may be written as $\bar{\Lambda}_{\mu'}{}^\rho \Lambda^{\nu'}{}_\rho = \delta^{\nu'}_{\mu'}$ and $\bar{\Lambda}_{\rho'}{}^\mu \Lambda^{\rho'}{}_\nu = \delta^\mu_\nu$, where $\delta^{\nu'}_{\mu'}$ and δ^μ_ν are Kronecker delta symbols.

   (b) Using the fact that the spacetime position x^μ is a 4-vector, find the partial derivatives $\partial x^\mu / \partial x^{\nu'}$ and $\partial x^{\mu'} / \partial x^\nu$ in terms of $\Lambda^{\mu'}{}_\nu$ and $\bar{\Lambda}_{\mu'}{}^\nu$.

   (c) If f is a Lorentz invariant function (meaning its value at a fixed spacetime point is the same in any frame — like the temperature), use the chain rule to show that
   $$\frac{\partial f}{\partial x^{\mu'}} = \bar{\Lambda}_{\mu'}{}^\nu \frac{\partial f}{\partial x^\nu} \ . \tag{4}$$
   In other words, you are showing that a partial derivative has the same transformation as a 4-vector with a lowered index. As a result, people will usually write $\partial_\mu f \equiv \partial f / \partial x^\mu$.

3. **The Relativistic Electromagnetic Field**

We won't prove it, but the electric and magnetic fields can be written as a relativistic tensor with two indices $F^{\mu\nu}$. This tensor is *antisymmetric*, meaning $F^{\nu\mu} = -F^{\mu\nu}$. The independent components are (here, $i = 1, 2, 3$ is a space index)

$$F^{oi} = E^i \ , \quad F^{12} = B^3 \ , \quad F^{13} = -B^2 \ , \quad F^{23} = B^1 \ . \tag{5}$$

Since $F^{\mu\nu}$ is antisymmetric, the diagonal components $F^{00} = F^{11} = F^{22} = F^{33} = 0$. (We have chosen a convenient system of units where the electric and magnetic field have the same dimension.)

(a) Consider two frames S and S' in standard configuration with each other. Show that

$$E^{3'} = \gamma\left(E^3 + \frac{v}{c}B^2\right) \quad \text{and} \quad B^{3'} = \gamma\left(B^3 - \frac{v}{c}E^2\right) \ . \tag{6}$$

Hint: Remember that the Lorentz transformation of a tensor transforms each index independently:

$$F^{\mu'\nu'} = \Lambda^{\mu'}{}_{\alpha}\Lambda^{\nu'}{}_{\beta}F^{\alpha\beta} \ . \tag{7}$$

(b) Calculating $F_{\mu\nu}F^{\mu\nu}$ and argue that $\vec{E}^2 - \vec{B}^2$ is a Lorentz invariant quantity.