PHYS-3301 Homework 7 Due 2 Nov 2011

This homework is due in class on the due date. If you wish to turn it in ahead of time, you may email a PDF or give a hardcopy to Dr. Frey.

1. **The Lorentz Group**

   In this problem, you will check some properties of the Lorentz group. We define the metric η as

$$\eta = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \tag{1}$$

   and the boost Λ_{tx} along x and the rotation Λ_{xy} in the xy plane (around the z axis) as follows:

$$\Lambda_{tx}(\phi) = \begin{bmatrix} \cosh\phi & -\sinh\phi & & \\ -\sinh\phi & \cosh\phi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} , \quad \Lambda_{xy}(\theta) = \begin{bmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{bmatrix} . \tag{2}$$

   Empty elements in the matrices above are zero.

   (a) Show that both Λ_{tx} and Λ_{xy} satisfy $\Lambda^T\eta\Lambda = \eta$.

   (b) Consider two successive boosts along x, $\Lambda_{tx}(\phi_1)$ and $\Lambda_{tx}(\phi_2)$. Show that these multiply to give a third boost $\Lambda_{tx}(\phi_3)$ and find ϕ_3. Using the relationship $v/c = \tanh\phi$ between velocity and rapidity ϕ, reproduce the velocity addition rule. *Hint:* You will need the angle-addition rules for hyperbolic trig functions.

   (c) Write down the Lorentz transformation matrix $\Lambda_{ty}(\phi)$ corresponding to a boost along the y direction. Then write a Taylor series for each of $\Lambda_{tx}(\phi_1)$, $\Lambda_{ty}(\phi_2)$, and $\Lambda_{xy}(\theta)$ to second order in ϕ_1, ϕ_2, θ around zero. Finally, show that

$$\Lambda_{ty}^{-1}(\phi_2)\Lambda_{tx}^{-1}(\phi_1)\Lambda_{ty}(\phi_2)\Lambda_{tx}(\phi_1) = \Lambda_{xy}(\theta = -\phi_1\phi_2) , \tag{3}$$

   still valid to second order in ϕ_1, ϕ_2, θ.

2. **4-Vectors and Changing Frames**

   (a) *Barton 7.2* Some component of the 4-vector $\tilde{a}$ is zero in every inertial reference frame. Show that $\tilde{a}$ is the zero vector.

   (b) If a 4-vector $\tilde{a}$ is timelike, show that there exists an inertial frame where the only nonzero component is a^0 (that is, the spatial part $\vec{a}$ is zero).

   Following based on Barton 7.5 In the following, $\tilde{a}$ and $\tilde{b}$ are two orthogonal 4-vectors $\tilde{a}\cdot\tilde{b} = 0$.

   (c) If $\tilde{a}$ and $\tilde{b}$ are both lightlike, prove that they are parallel ($\tilde{a} \propto \tilde{b}$).

   (d) Suppose $\tilde{a}$ is timelike. What is the component b^0 in the inertial frame where $\vec{a} = 0$? Assuming $\tilde{b} \neq 0$, is $\tilde{b}$ spacelike, timelike, or lightlike?

   (e) Suppose $\tilde{a}^2 = -\tilde{b}^2$. Is the 4-vector $\tilde{a} + \tilde{b}$ spacelike, timelike, or lightlike?