

The metric dimension of circulant graphs and Cayley hypergraphs

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Abstract

Let $G = (V, E)$ be a connected graph (or hypergraph) and let $d(x, y)$ denote the distance between vertices $x, y \in V(G)$. A subset $W \subseteq V(G)$ is called a resolving set for G if for every pair of distinct vertices $x, y \in V(G)$, there is $w \in W$ such that $d(x, w) \neq d(y, w)$. The minimum cardinality of a resolving set for G is called the metric dimension of G , denoted by $\beta(G)$.

In this paper we determine the exact metric dimension of the circulant graphs $C_n(1, 2)$ and $C_n(1, 2, 3)$ for all n , extending previous results due to Javaid and Rahim (2008) and Imran, Baig, Bokhary and Javaid (2011). In particular, we show that $\beta(C_n(1, 2)) = 4$ if $n \equiv 1 \pmod{4}$ and $\beta(C_n(1, 2)) = 3$ otherwise. We also show that $\beta(C_n(1, 2, 3)) = 5$ if $n \equiv 1 \pmod{6}$ and $\beta(C_n(1, 2, 3)) = 4$ otherwise. In addition, we bound the metric dimension of Cayley hypergraphs on finite Abelian groups with the canonical set of generators, and we show that the metric dimension of these hypergraphs is related to the metric dimension of a Cartesian product of circulant graphs.

Key words: Metric dimension, Circulant graphs, Cayley hypergraphs

AMS Subject Classification Codes: 05C12, 05C65, 05C25, 05E18.

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1 Introduction

1.1 Definitions

A *hypergraph* is a pair (V, E) in which V is a finite set of *vertices* and E is a collection of subsets of V called *edges*. A hypergraph $H = (V, E)$ is called *k-uniform* (or a *k-hypergraph*) if E is a set of k -element subsets of V . A *graph* is a hypergraph in which the cardinality of every edge is at most 2. A *path* of length k in a hypergraph (V, E) is a sequence $(v_1, e_1, v_2, \dots, v_k, e_k, v_{k+1})$ in which $v_i \in V$ for $i = 1, 2, \dots, k+1$, $e_i \in E$ for $i = 1, 2, \dots, k$, $\{v_i, v_{i+1}\} \subseteq e_i$ for $i = 1, 2, \dots, k$, and $v_i \neq v_j$ and $e_i \neq e_j$ for $i \neq j$. A hypergraph is *connected* if there is a path between every pair of vertices. The *distance* between two vertices in a hypergraph is the length of a shortest path between them. A vertex x in a hypergraph H is said to *resolve* a pair u, v of vertices of H if the distance from u to x does not equal the distance from v to x . A set W of vertices of H is a *resolving set* for H if every pair of vertices of H is resolved by some vertex of W . The smallest cardinality of a resolving set for H is called the *metric dimension* of H , and is denoted by $\beta(H)$.

The metric dimension appears to be related to both local and global symmetry in graphs and hypergraphs. Consequently we are motivated to examine the metric dimension of Cayley hypergraphs as these are classes of vertex transitive hypergraphs for which degrees of symmetry may vary. Cayley hypergraphs have the added advantage that distances between pairs of vertices can be described algebraically, thus lending themselves more readily to the use of algebraic tools when computing distance related invariants. Let Γ be a group, let $\Omega \subseteq \Gamma \setminus \{1\}$, and let t be an integer such that $2 \leq t \leq \max\{|\omega| : \omega \in \Omega\}$. The *t-Cayley hypergraph* of Γ over Ω , denoted $H = t\text{-Cay}[\Gamma : \Omega]$, is the hypergraph with vertex set Γ in which a subset $S \subseteq \Gamma$ is in $E(H)$ if and only if there is $x \in \Gamma$ and $\omega \in \Omega$ such that $S = \{x\omega^i : 0 \leq i \leq t-1\}$. Note that a 2-Cayley hypergraph is a Cayley graph. This definition is due to Buratti [2], and is a subclass of the more general Cayley hypergraphs, or *group hypergraphs* which were defined by Shee in [12].

In this paper we investigate the metric dimension of Cayley hypergraphs on finite Abelian groups, which we will see in Section 3 are related to the metric dimension of Cartesian products of circulant graphs, which we now define. For positive integers t and n , the *circulant graph* $C_n(1, 2, \dots, t)$ is the simple graph with vertex set \mathbb{Z}_n , the integers modulo n , in which distinct vertices i and j are adjacent if and only if $|i - j| \pmod{n} \leq t$. Thus vertex i is adjacent to the vertices $i - t, i - t + 1, \dots, i - 1, i + 1, \dots, i + t - 1, i + t \pmod{n}$ in $C_n(1, 2, \dots, t)$. (See Figure 1.) Observe that the distance

between two vertices i and j in $G = C_n(1, 2, \dots, t)$ is given by

$$d_G(i, j) = \begin{cases} \left\lceil \frac{|i-j|}{t} \right\rceil, & |i-j| \leq \frac{n}{2} \\ \left\lceil \frac{-|i-j|(\bmod n)}{t} \right\rceil, & |i-j| > \frac{n}{2} \end{cases}.$$

The *outer cycle* of the circulant graph $G = C_n(1, 2, \dots, t)$ is a spanning subgraph of G in which vertex i is adjacent to exactly the vertices $i+1$ and $i-1$.

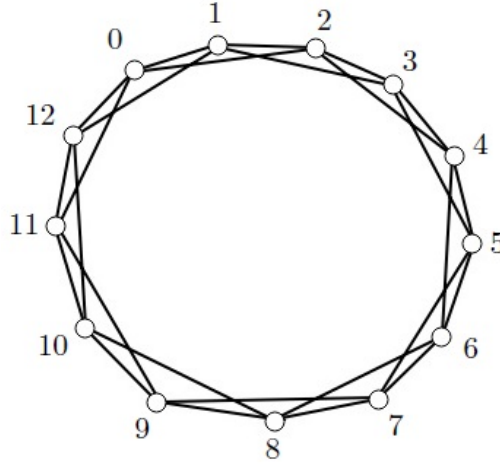


Figure 1: $C_{13}(1, 2)$

The *Cartesian product* of graphs G_1 and G_2 , denoted by $G_1 \square G_2$, is the graph with vertex set $V(G_1) \times V(G_2) := \{(x, y) : x \in V(G_1), y \in V(G_2)\}$, in which (x, y) is adjacent to (x', y') whenever $x = x'$ and $yy' \in E(G_2)$, or $y = y'$ and $xx' \in E(G_1)$. Observe that if G_1 and G_2 are connected graphs, then $G_1 \square G_2$ is connected. Assuming that isomorphic graphs are equal, the Cartesian product is associative, so $G_1 \square G_2 \square \dots \square G_d$ is well-defined for graphs G_1, G_2, \dots, G_d . Moreover, for two vertices $\mathbf{x} = (x_1, x_2, \dots, x_d)$ and $\mathbf{y} = (y_1, y_2, \dots, y_d)$ of the graph $G = G_1 \square G_2 \square \dots \square G_d$, the distance $d_G(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^d d_{G_i}(x_i, y_i)$.

1.2 History and layout of the paper

Motivated by the problem of efficiently locating a moving point or intruder in a network, the concept of the metric dimension of a graph (2-hypergraph) was first introduced by Slater [13, 14], and independently by Harary and Melter [5]. Slater referred to the metric dimension of a graph as its location

number and motivated the study of this invariant by its application to the placement of a minimum number of sonar/loran detecting devices in a network so that the position of every vertex in the network can be uniquely described in terms of its distances to the devices in the set. Khuller et al [9] studied the metric dimension as an application to the navigation of robots in a graph space. A resolving set for a graph corresponds to a set of landmark nodes in the graph, and it is assumed that a robot navigating a graph can sense the distance to each of these landmarks and hence uniquely determine its location in the graph. They gave a construction to show that the problem of determining the metric dimension of a graph is NP-hard. The problem received renewed attention in [4] as it also has applications to a problem in pharmaceutical chemistry. The metric dimension of a graph is related to several other well studied graph invariants such as the *determining number* (the base size of its automorphism group), and a good survey of these invariants and their relation to one another was written by Bailey and Cameron in 2011 [1]. Since the problem of determining the metric dimension of a graph is known to be NP-hard, researchers have focussed on computing or bounding the metric dimension of certain classes of graphs, and on constructing resolving sets for these classes of graphs.

Due to the fact that metric dimension has applications in network discovery and verification, combinatorial optimization, chemistry, and many other areas, this graph parameter has received a great deal of attention from researchers recently. In particular, Javaid et al [8] and Imran et al [7] have studied the metric dimension of circulant graphs, and obtained the following results.

Proposition 1 [8] *For the circulant graphs $C_n(1, 2)$, we have*

- (1) $\beta(C_n(1, 2)) = 3$ for $n \equiv 0, 2, 3 \pmod{4}$, and
- (2) $\beta(C_n(1, 2)) \leq 4$ otherwise.

Proposition 2 [7] *For the circulant graphs $C_n(1, 2, 3)$, we have*

- (1) $\beta(C_n(1, 2, 3)) = 4$ for $n \equiv 2, 3, 4, 5 \pmod{6}$ and $n \geq 14$.
- (2) $\beta(C_n(1, 2, 3)) \leq 5$ for $n \equiv 0 \pmod{6}$ and $n \geq 12$.
- (3) $\beta(C_n(1, 2, 3)) \leq 6$ for $n \equiv 1 \pmod{6}$ and $n \geq 13$.

In Section 2 we will extend these results to find the exact metric dimension of both $C_n(1, 2)$ and $C_n(1, 2, 3)$ for all n (see Theorem 8).

Cáceres et al [3], and independently Peters-Franzen and Oellermann [11], have studied the metric dimension of Cartesian products of graphs, and they obtained the following result.

Proposition 3 [3, 11] *Let G be a graph and let $n \geq m \geq 3$. Then*

$$\beta(G) \leq \beta(G \square C_m) \leq \begin{cases} \beta(G) + 1 & \text{if } m \text{ is odd} \\ \beta(G) + 2 & \text{if } m \text{ is even} \end{cases},$$

$$\beta(C_m \square C_n) = \begin{cases} 3, & \text{if } m \text{ or } n \text{ is odd} \\ 4, & \text{if } m \text{ and } n \text{ are both even} \end{cases},$$

and

$$\beta(C_n \square K_2) = \begin{cases} 2, & \text{if } n \text{ is odd} \\ 3, & \text{if } n \text{ is even} \end{cases}.$$

Cáceres et al [3] also bounded the metric dimension of a Cartesian product of graphs in terms of another related parameter called the doubly resolving number, which we now define. Two vertices v and w of a graph G are *doubly resolved* by $x, y \in V(G)$ if $d(v, x) - d(w, x) \neq d(v, y) - d(w, y)$. A set $S \subseteq V(G)$ *doubly resolves* G , and is a *doubly resolving set* for G , if every pair of distinct vertices of G are doubly resolved by two vertices of S . The *doubly resolving number* of G , denoted by $\psi(G)$, is the minimum cardinality of a doubly resolving set for G . Every graph with at least two vertices has a doubly resolving set, so $\psi(G)$ is well-defined for $G \neq K_1$ (the trivial graph). Note that if vertices x, y doubly resolve vertices v, w , then either $d(v, x) - d(w, x) \neq 0$ or $d(v, y) - d(w, y) \neq 0$, and so at least one of x or y resolves the pair v, w . Thus a doubly resolving set is also a resolving set, and consequently $\beta(G) \leq \psi(G)$ for every nontrivial graph G . Also, it was shown in [3] that $\psi(G) \leq |V(G)| - 1$ for any connected graph G with at least three vertices, since $V(G) \setminus \{x\}$ doubly resolves G for any vertex x of degree at least 2. We have the following result.

Proposition 4 [3] *For all graphs G_1 and $G_2 \neq K_1$,*

$$\max\{\beta(G_1), \beta(G_2)\} \leq \beta(G_1 \square G_2) \leq \beta(G_1) + \psi(G_2) - 1.$$

For $n \geq 3$, any set of $n - 1$ vertices of the complete graph K_n is a doubly resolving set, and clearly no smaller subset of vertices of K_n is doubly resolving. It follows that $\psi(K_n) = n - 1$ for $n \geq 3$. Thus Proposition 4 implies the following result.

Corollary 1 *For a graph G and $n \geq 3$, $\beta(G) \leq \beta(G \square K_n) \leq \beta(G) + n - 2$.*

Cáceres et al also proved the following.

Proposition 5 [3] *For $n \geq m \geq 1$ we have*

$$\beta(K_n \square K_m) = \begin{cases} \lfloor \frac{2}{3}(n + m - 1) \rfloor, & \text{if } m \leq n \leq 2m - 1 \\ n - 1, & \text{if } n \geq 2m - 1 \end{cases}.$$

Earlier in [4], Chartrand et al obtained the following result.

Proposition 6 [4] *For every connected graph G , $\beta(G) \leq \beta(G \square K_2) \leq \beta(G) + 1$.*

In Section 3, we will show that $\beta(C_n(1,2) \square K_2) = \beta(C_n(1,2))$ (See Theorem 12).

More recently, Manrique and Arumugam [10] have initiated a study of the metric dimension of hypergraphs. They showed that the metric dimension of a hypergraph is equal to the metric dimension of a related graph called its 2-section. For a hypergraph $H = (V, E)$ and a positive integer k , the k -section of H is the hypergraph $H_k = (V, E_k)$, where for every set $e \subseteq V$, $e \in E_k$ if either $|e| \leq k$ and $e \in E$ or $|e| = k$ and $e \subseteq f$ for some $f \in E$. Note that the 2-section H_2 of a hypergraph H is a graph. The following result was proved in [10].

Proposition 7 *For every hypergraph H and every positive integer k , a subset $W \subseteq V$ is resolving in H if and only if it is resolving in H_k . Therefore $\beta(H) = \beta(H_k)$.*

In Section 3 we will show that the 2-section of a t -Cayley hypergraph on a finite Abelian group with the canonical set of generators is a Cartesian product of circulant graphs, and we will use this fact to bound the metric dimension of these group hypergraphs.

2 Metric dimension of circulant graphs

In this section we will determine the metric dimension of the circulant graph $C_n(1, 2, \dots, t)$ for $t \in \{2, 3\}$, for all n . The main technique here is the use of special hypergraphs related to a graph, called resolving hypergraphs, which we now define.

Definition 1 *For a graph G and a set of vertices $W \in G$, we define the resolving hypergraph of G with respect to W is the hypergraph with vertex set $V(G)$, and hyperedges W_{i_d} , where W_{i_d} contains all vertices at distance d from w_i in G , for $1 \leq d \leq k$ (Figure 3), where $k = \text{diam}(G)$. We denote this hypergraph by $R_W(G)$, or simply R_W if G is understood.*

Remark 1 *The diameter of $G = C_n(1, 2, \dots, t)$ is the quotient upon division of n by $2t$. Note that when the number of vertices of G is given as $n = (2t)k + r$, for $0 \leq r \leq 2t - 1$, the diameter of G is k .*

Remark 2 Note that W is a resolving set of G if, and only if, each vertex has a unique edge neighbourhood in $R_W(G)$. This is because two vertices with the same edge neighbourhood in $R_W(G)$ are unresolved by W . In this way, we see that $W = \{w_1, w_2, w_3, w_4\}$ resolves $C_{13}(1, 2)$ (Figure 4).

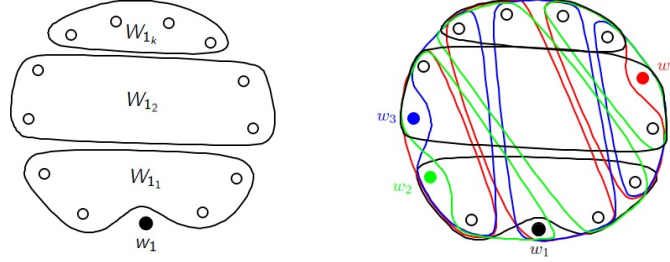


Figure 2: $R_{\{w_1\}}(C_{13}(1, 2))$ Figure 3: $R_{\{w_1, w_2, w_3, w_4\}}(C_{13}(1, 2))$

We now look at how a resolving hypergraph can be used to prove results on the metric dimension of circulant graphs.

Lemma 1 No clique of three vertices in a graph G can be pairwise resolved by a single vertex.

Proof: Let $X = \{x, y, z\}$ be a clique of three vertices in G . Suppose x and y are resolved by some vertex v in G . Since x and y are adjacent, $d(v, x) = d(v, y) \pm 1$. Then, either $d(v, z) = d(v, x)$, or $d(v, z) = d(v, y)$ and v does not resolve X .

Theorem 1 For $n \equiv 1 \pmod{4}$, $\beta(C_n(1, 2)) \geq 4$.

Proof: Let $G = C_n(1, 2)$ and $n = 4k + 1$ for some integer $k \geq 2$. Note that k is the diameter of G . Suppose, to the contrary, that a metric basis W exists for G such that $|W| = 3$. Say $W = \{w_1, w_2, w_3\}$. By the symmetry of G the choice of w_1 is arbitrary, so we choose vertex 0 as w_1 . We now consider three cases for the choice of w_2 .

Case 1: w_2 is adjacent to w_1 in the outer cycle of G (In this case, $w_2 = 1$ or $w_2 = n - 1$. Say the latter). This leaves vertices $\{\frac{n+1}{2} - 1, \frac{n+1}{2}, \frac{n+1}{2} + 1\}$ all contained in both W_{1k} and W_{2k} in $R_{\{w_1, w_2\}}$ (Figure 4). Hence, vertices in the set $\{\frac{n+1}{2} - 1, \frac{n+1}{2}, \frac{n+1}{2} + 1\}$ are pairwise unresolved by $\{w_1, w_2\}$ in G . By Lemma 1, no choice of w_3 will resolves all of these pairs simultaneously. This gives the desired contradiction. Hence, w_2 cannot be adjacent to w_1 in the outer cycle of G . Furthermore, since the order in which vertices for

W are chosen does not matter, no two vertices of W may be adjacent in the outer cycle of G .

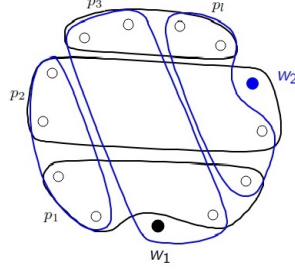
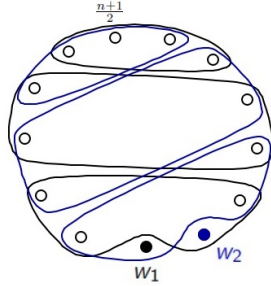


Figure 4: $R_{\{w_1, w_2\}}(C_{13}(1, 2))$ Figure 5: $R_{\{w_1, w_2\}}(C_{13}(1, 2))$

Case 2: The distance between w_1 and w_2 in the outer cycle of G is even. Say $w_2 = n - 2m$ for $1 \leq m \leq k$. This leaves a sequence of disjoint pairs of vertices with shared edge neighbourhoods in $R_{\{w_1, w_2\}}$. These pairs are $\{1, 2\}, \{3, 4\}, \{5, 6\}, \dots, \{n - 2m - 2, n - 2m - 1\}$. Label these unresolved pairs p_1, p_2, \dots, p_l respectively (Figure 5), where $l = \frac{n-2m-1}{2}$. By Case 1, w_3 cannot be adjacent to w_1 or w_2 in the outer cycle of G . If w_3 is a vertex on the shortest (even length) path between w_1 and w_2 on the outer cycle of G , and is not adjacent to w_1 or w_2 , then some p_i is contained in W_{3_k} (Figure 6). Hence G is not resolved by W . If w_3 is a vertex on the odd length path between w_1 and w_2 on the outer cycle, and w_3 is not adjacent to w_1 or w_2 on the outer cycle, then w_3 belongs to some unresolved pair p_i . In this case, exactly one of p_{i-1} or p_{i+1} is contained in W_{3_1} , again leaving G unresolved by W (Figure 7). This gives the desired contradiction. We conclude that w_2 cannot be at an even distance from w_1 in the outer cycle of G . Furthermore, no two vertices in W can be at an even distance from each other in the outer cycle of G .

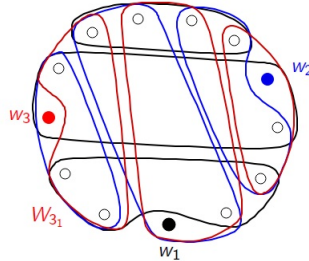
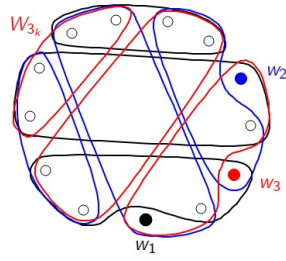


Figure 6: $R_{\{w_1, w_2, w_3\}}(C_{13}(1, 2))$ Figure 7: $R_{\{w_1, w_2, w_3\}}(C_{13}(1, 2))$

Case 3: The distance between w_1 and w_2 in the outer cycle of G is odd. Let $w_2 = n - 2m - 1$ for $1 \leq m \leq k - 1$. In this case, the vertices between w_1 and w_2 are divided into a sequence of unresolved pairs, $\{n - 1, n - 2\}, \{n - 3, n - 4\}, \dots, \{n - 2m, n - 2m + 1\}$. Also, the middle two vertices of W_{1k} , namely $\{\frac{n+1}{2}, \frac{n+1}{2} - 1\}$, and the middle two vertices of W_{2k} , namely $\{\frac{n-4m-1}{2}, \frac{n-4m-1}{2} - 1\}$, are unresolved pairs. Denote these p_{w_1} and p_{w_2} , respectively (Figure 8). Note that p_{w_1} is contained in W_{2n-k} , and p_{w_2} is contained in W_{1n-k} .

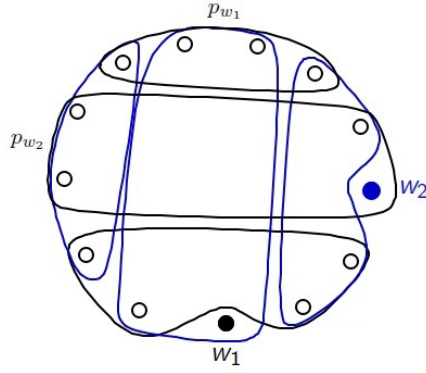


Figure 8: $R_{\{w_1, w_2\}}(C_{13}(1, 2))$

Note that there is always an odd number of vertices between p_{w_1} and p_{w_2} in the outer cycle of G . This is because G has an odd number of vertices, but there is an even number of vertices between w_1 and w_2 , and the same number between w_1 and p_{w_2} as there are between w_2 and p_{w_1} . Note that only vertices at odd distance from w_1 can resolve the sequence of pairs $\{n - 1, n - 2\}, \{n - 3, n - 4\}, \dots, \{n - 2m, n - 2m + 1\}$ left unresolved by w_1 and w_2 , but the only vertex at odd distance from w_1 which also resolves the pair p_{w_1} is the vertex in p_{w_1} closer to w_1 , which doesn't resolve the pair p_{w_2} . Hence there are no valid choices for w_3 in this case. Thus $\beta(G) \geq 4$.

□

Theorem 2 For $n \equiv 0 \pmod{6}$, $n \geq 12$, $\beta((C_n(1, 2, 3))) \leq 4$.

Proof: Let $G = C_n(1, 2, 3)$ and $n = 6k$ for some integer $k \geq 2$. Note that $k = \text{diam}(G)$. We claim that $W = \{0, 2, \frac{n}{2}, \frac{n}{2} + 2\}$ is a metric basis of G . Observe that the following sets of vertices share edge neighbourhoods in

$R_{\{0,2\}}$:
 $\{1, 3, n-1\}, \{4, 5\}, \{n-2, n-3\}, \{6, n-4\}, \{7, 8\}, \{n-5, n-6\}, \dots,$
 $\{\frac{n}{2}+5, \frac{n}{2}-3\}, \{\frac{n}{2}-1, \frac{n}{2}-2\}, \{\frac{n}{2}+4, \frac{n}{2}+3\}, \{\frac{n}{2}+2, \frac{n}{2}+1, \frac{n}{2}\}$ (Figure 9).

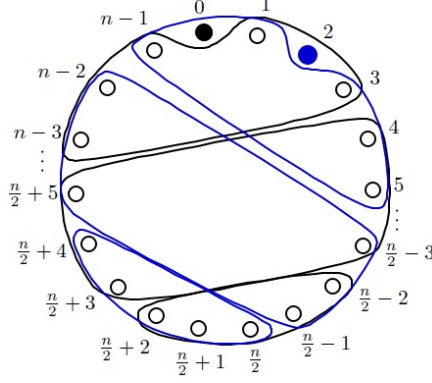


Figure 9: $R_{\{0,2\}}(C_{18}(1, 2, 3))$

We now list the vertices sharing edge neighbourhoods in $R_{\{\frac{n}{2}, \frac{n}{2}+2\}}$. They are exactly the following:
 $\{\frac{n}{2}-1, \frac{n}{2}+1, \frac{n}{2}+3\}, \{\frac{n}{2}+4, \frac{n}{2}+5\}, \{\frac{n}{2}-2, \frac{n}{2}-3\},$
 $\{\frac{n}{2}+6, \frac{n}{2}-4\}, \dots, \{n-4, n-5\}, \{6, 7\}, \{5, n-3\}, \{n-1, n-2\}, \{3, 4\},$
 $\{0, 1, 2\}$. Observe that no pair is left unresolved by both $\{0, 2\}$ and $\{\frac{n}{2}, \frac{n}{2}+2\}$ in G . Thus, W is a metric basis of G , and $\beta(G) \leq 4$.

□

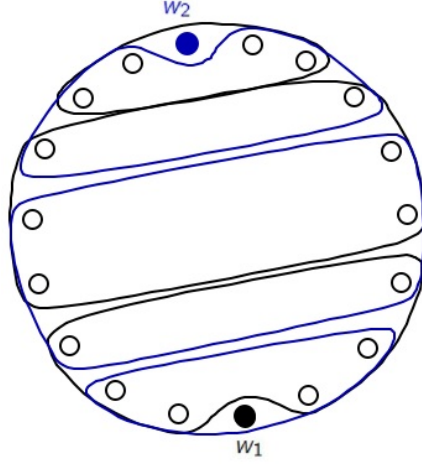


Figure 10: $R_{\{w_1, w_2\}}(C_{18}(1, 2, 3))$

Theorem 3 For $n \equiv 0 \pmod{6}$, $n \geq 12$, $\beta(C_n(1, 2, 3)) \geq 4$.

Proof: Let $G = C_n(1, 2, 3)$ and $n = 6k$ for some integer $k \geq 2$. Note that $k = \text{diam}(G)$. Suppose that a metric basis $W \in V(G)$ exists such that $|W| = 3$. Say $W = \{w_1, w_2, w_3\}$. By the symmetry of G , we may take $w_1 = 0$. Now, consider all the possible choices for w_2 . By Lemma 1, w_2 must be chosen in a way such that no clique of three vertices are pairwise unresolved in G . The only choice of w_2 satisfying this condition is $\frac{n}{2}$ (Figure 10), as any other choice for w_2 leaves a clique of three pairwise unresolved vertices in G contained in W_{1_k} . The following are the sets of vertices sharing edge neighbourhoods in $R_{\{w_1, w_2\}}$: $\{n-2, n-1, 1, 2\}$, $\{n-3, 3\}$, $\{n-5, n-4, 4, 5\}$, \dots , $\{\frac{n}{2}-3, \frac{n}{2}+3\}$, $\{\frac{n}{2}-2, \frac{n}{2}-1, \frac{n}{2}+1, \frac{n}{2}+2\}$. In this case, any choice of w_3 will leave at least one unresolved pair contained in W_{3_k} , giving the desired contradiction. Hence, $\beta(G) \geq 4$. □

The following theorem follows immediately from Theorem 2 and Theorem 3.

Theorem 4 For $n \equiv 0 \pmod{6}$, $n \geq 12$, $\beta(C_n(1, 2, 3)) = 4$.

Theorem 5 For $n \equiv 1 \pmod{6}$, $\beta(C_n(1, 2, 3)) \leq 5$.

Proof: Let $G = C_n(1, 2, 3)$. We claim that $W = \{0, 1, 5, 6, \frac{n+1}{2} + 1\}$ is a resolving set for G . To show this, we first observe the sets of vertices

sharing edge neighbourhoods in $R_{\{0,1\}}$. These are exactly the following:

$$\begin{aligned} & \{2, 3, n-1, n-2\}, \{5, 6, n-4, n-5\}, \{8, 9, n-7, n-8\}, \dots, \\ & \left\{ \frac{n-1}{2} - 7, \frac{n-1}{2} - 8, \frac{n-1}{2} + 7, \frac{n-1}{2} + 8 \right\}, \\ & \left\{ \frac{n+1}{2} - 4, \frac{n+1}{2} - 5, \frac{n+1}{2} + 4, \frac{n+1}{2} + 5 \right\}, \\ & \left\{ \frac{n+1}{2} - 2, \frac{n+1}{2} - 1, \frac{n+1}{2}, \frac{n+1}{2} + 1, \frac{n+1}{2} + 2 \right\} \text{ (Figure 11)}. \end{aligned}$$

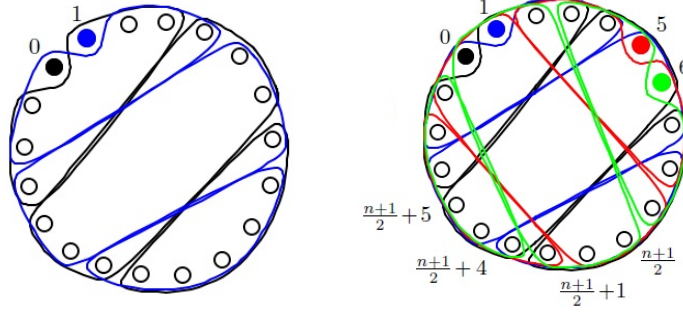


Figure 11: $R_{\{0,1\}}(C_{19}(1,2,3))$ Figure 12: $R_{\{0,1,5,6\}}(C_{19}(1,2,3))$

Similarly, the sets of vertices sharing edge neighbourhoods in $R_{\{5,6\}}$ are exactly

$$\begin{aligned} & \{3, 4, 7, 8\}, \{0, 1, 10, 11\}, \{n-2, n-3, 13, 14\}, \dots, \\ & \left\{ \frac{n+1}{2} - 2, \frac{n+1}{2} - 3, \frac{n+1}{2} + 12, \frac{n+1}{2} + 13 \right\}, \\ & \left\{ \frac{n+1}{2}, \frac{n+1}{2} + 1, \frac{n+1}{2} + 9, \frac{n+1}{2} + 10 \right\}, \\ & \left\{ \frac{n+1}{2} + 3, \frac{n+1}{2} + 4, \frac{n+1}{2} + 5, \frac{n+1}{2} + 6, \frac{n-1}{2} + 7 \right\}. \end{aligned}$$

The only pairs sharing edge neighbourhoods in $R_{\{0,1,5,6\}}$ are $\left\{ \frac{n+1}{2}, \frac{n+1}{2} + 1 \right\}$ and $\left\{ \frac{n+1}{2} + 4, \frac{n+1}{2} + 5 \right\}$ (Figure 12). Since $d\left(\frac{n+1}{2} + 1, \frac{n+1}{2} + 4\right) = 1 \neq 2 = d\left(\frac{n+1}{2} + 1, \frac{n+1}{2} + 5\right)$, $\frac{n+1}{2} + 1$ resolves both remaining pairs. Hence W resolves G , and $\beta(G) \leq 5$.

□

Theorem 6 For $n \equiv 1 \pmod{6}$, $\beta(C_n(1,2,3)) \geq 5$.

Lemma 2 Suppose $n \not\equiv 3 \pmod{6}$. Then no clique of four vertices in a circulant graph $G = C_n(1,2,3)$ can be pairwise resolved by any two vertices.

Proof: Let $X = \{w, x, y, z\}$ be a clique of four vertices in G . We may assume that the sequence w, x, y, z are consecutive vertices which form an increasing path on the outer cycle of G . Choose some vertex u in G . By Lemma 1, if u leaves a clique of three pairwise unresolved vertices in X , then no choice of a second vertex, v , resolves X . Thus, u must be chosen such that u has distance d to two of the vertices of X and distance $d+1$ to

the other two vertices of X . Given that the sequence w, x, y, z are consecutive vertices which form an increasing path on the outer cycle of G , there are two cases to consider.

Case 1: $d(u, w) = d(u, z) = d$ and $d(u, x) = d(u, y) = d + 1$. In this case, u is antipodal to the clique formed by w, x, y, z on the outer cycle. That is, $|w - u| \leq n/2$ and $|x - u| \leq n/2$ while $|y - u| > n/2$ and $|z - u| > n/2$. Since $d(u, w) = d(u, x) - 1$ and $d(u, z) = d(u, y) - 1$, the distance formula implies that $|w - u| \equiv 0 \pmod{3}$ and $|u - z| \equiv 0 \pmod{3}$, which implies that $n \equiv 3 \pmod{6}$, giving a contradiction.

Case 2: $d(u, w) = d(u, x) = d$ and $d(u, y) = d(u, z) = d + 1$. Suppose that v is chosen such that v resolves w and x . Then $d(v, w) = d(v, x) \pm 1$. It follows that $d(v, y) = d(v, z) = d(v, x)$, except in the case where v is antipodal to the clique formed by w, x, y, z on the outer cycle. In that case, one can use a similar argument to that in Case 1 to show that $n \equiv 3 \pmod{6}$, giving a contradiction. Hence the vertices of X are not resolved by u and v .

Proof of Theorem 6: Let $G = C_n(1, 2, 3)$, $n = 6k + 1$ for $k \geq 2$ (Note that $k = \text{diam}(G)$). Suppose a resolving set $W \in V(G)$ exists, such that $|W| = 4$. Say $W = \{w_1, w_2, w_3, w_4\}$. By the symmetry of G , we may take $w_1 = 0$. If $w_2 = 1$ or $w_2 = n - 1$ (say the latter), then the vertices $\{\frac{n-1}{2} - 2, \frac{n-1}{2} - 1, \frac{n-1}{2}, \frac{n-1}{2} + 1, \frac{n-1}{2} + 2\}$ share the edge neighbourhood $\{W_{1_k}, W_{2_k}\}$ in $R_{\{w_1, w_2\}}$. Since a clique of four vertices is contained in this set, no choice of w_3 and w_4 completely resolve G by Lemma 2. If $w_2 = 2$ or $w_2 = n - 2$ (again, say the latter), then the vertices in the set $\{\frac{n-1}{2} - 2, \frac{n-1}{2} - 1, \frac{n-1}{2}, \frac{n-1}{2} + 1\}$ share the edge neighbourhood $\{W_{1_k}, W_{2_k}\}$ in $R_{\{w_1, w_2\}}$. Since this is a clique of four vertices, as previously, no choice of w_3 and w_4 completely resolve G by Lemma 2. Hence, since the order in which vertices are chosen for a resolving set does not matter, no two vertices in W may be at a distance of one or two from each other in the outer cycle of G . We now consider general cases for the choice of w_2 . Let d denote the the distance from w_1 to w_2 in the outer cycle of G .

Case 1: $d \equiv 0 \pmod{3}$. By the symmetry of G , suppose $w_2 = 3m$ for $1 \leq m \leq k$. The following are sets of vertices that share edge neighbourhoods in $R_{\{w_1, w_2\}}$: $\{1, 2\}, \{4, 5\}, \dots, \{3m - 2, 3m - 1\}$ and $\{3m + 1, 3m + 2, 3m + 3\}, \{3m + 4, 3m + 5, 3m + 6\}, \dots, \{n - 3, n - 2, n - 1\}$. Denote these sets of three vertices p_1, p_2, \dots, p_l respectively, where $l = \frac{n - w_2 - 1}{3}$. By Lemma 1, w_3 must be chosen such that no p_i is contained in W_{3_j} , for $1 \leq j \leq k$. If w_3 lies between w_1 and w_2 , then some p_i is contained in W_{3_k} . If w_3

is the first vertex (the vertex with the smallest index) in some p_i , $i > 1$, then W_{3_1} contains p_{i-1} . Similarly, if w_3 is the last vertex (the vertex with the greatest index) in some p_i , $i < l$, then W_{3_1} contains p_{i+1} . Since no two vertices in W may be separated by zero or one vertices in the outer cycle of G , w_3 cannot be either of the first two vertices in p_1 , or the last two vertices in p_l . Figure 15 shows all invalid choices for w_3 crossed out. Note that the only valid choices for w_3 are the middle vertices in p_2, p_3, \dots, p_{l-1} . Denote these v_1, v_2, \dots, v_{l-2} . Since the order of which w_3 and w_4 are chosen is irrelevant, these are also the only choices for w_4 . Observe that $d(v_i, 3m+2) = d(v_i, 3m+3) = i$ for $1 \leq i \leq l-1$. Hence, the pair $\{3m+2, 3m+3\}$ will always be left unresolved by W , giving the desired contradiction.

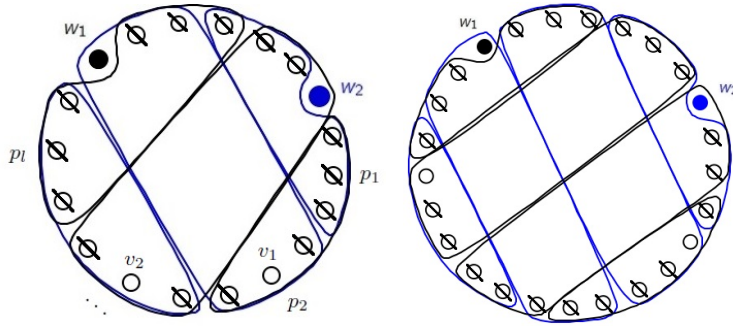


Figure 13: $R_{\{w_1, w_2\}}(C_{19}(1, 2, 3))$ Figure 14: $R_{\{w_1, w_2\}}(C_{25}(1, 2, 3))$

Case 2: $d \equiv 1 \pmod{3}$. By the symmetry of G , suppose $w_2 = 3m+1$ for $1 \leq m \leq k-1$. The following are sets of vertices that share edge neighbourhoods in $R_{\{w_1, w_2\}}$ (although, they are not necessarily all of them). $\{1, 2, 3\}, \{4, 5, 6\}, \dots, \{3m-2, 3m-1, 3m\}$. These are the vertices between w_1 and w_2 . There are also $\{3m+2, 3m+3\}, \{3m+5, 3m+6\}, \dots, \{3k-4, 3k-3\}, \{3k-1, 3k, 3k+1\}$ and $\{n-1, n-2\}, \{n-4, n-5\}, \dots, \{3k+3m+6, 3k+3m+5\}, \{3k+3m+3, 3k+3m+2, 3k+3m+1\}$. We now look at conditions for choosing the two remaining vertices. By Lemma 1, the vertex w_3 must be chosen in a way that leaves no clique of three unresolved vertices in G . The vertex w_3 (or w_4) cannot be at a distance 1, 2, or $3j$, for $1 \leq j \leq k$, from w_1 or w_2 in the outer cycle of G (by case 1). This leaves the vertices $\{5, 8, \dots, 3m-4\}, \{3m+5, 3m+8, \dots, 3k-1\}$ and $\{n-4, n-7, \dots, 3k+3m+3\}$ as the only valid choices for w_3 and w_4 (Figure 14). Note that the distance between vertices in the same set on the outer cycle of G is a multiple of 3. The pair $\{3k, 3k+1\}$ is not resolved by

any of these vertices, since

$$\begin{aligned} d(3k, 5) = d(3k+1, 5) = k-1, d(3k, 8) = d(3k+1, 8) = k-2, \dots, d(3k, 3m-4) = d(3k+1, 3m-4) = 3, \\ d(3k, 3m+5) = d(3k+1, 3m+5) = k-m-1, d(3k, 3m+8) = d(3k+1, 3m+8) = k-m-2, \dots, d(3k, 3k-1) = d(3k+1, 3k-1) = 1, \text{ and} \\ d(3k, n-4) = d(3k+1, n-4) = k-1, d(3k, n-7) = d(3k+1, n-7) = k-2, d(3k, 3k+3m+3) = d(3k+1, 3k+3m+3) = m+1. \end{aligned}$$

Hence, we have the desired contradiction that W does not resolve G .

Case 3: $d \equiv 2 \pmod{3}$. By the symmetry of G , suppose $w_2 = 3m+2$ for $1 \leq m \leq k-1$. By the previous two cases, w_3 (or w_4) cannot be at a distance $2, 3i$, or $3j+1$, for $0 \leq j \leq k-1$, $0 \leq i \leq k$, from w_1 or w_2 in the outer cycle of G . Hence, w_3 (and w_4) must be at distances equivalent to 2 modulo 3 from both w_1 and w_2 in the outer cycle of G . We claim that there are no such vertices in G . Note that the vertices between w_2 and $\frac{n-1}{2}$ at a distance $3l+2$ from w_2 in the outer cycle of G , for some integer l , are at distance $(3l+2)+(3m+2) = 3(l+m+1)+1$ from w_1 in the outer cycle of G . Thus, these vertices cannot be chosen for w_3 . By the symmetry of G , the vertices from $3m+2+\frac{n+1}{2}$ to $n-1$ may also not be chosen. A vertex between w_1 and w_2 at distance $3l+2$ from w_1 in the outer cycle of G is at distance $(3m+2)-(3l+2) = 3(m-l)$ from w_2 in the outer cycle of G . Finally, a vertex between $\frac{n-1}{2}$ and $3m+2+\frac{n+1}{2}$ at distance $3l+2$ from w_2 in the outer cycle of G is at distance $n-(3m+2)-(3l+2) = 6k+1-3m-3l-4 = 3(2k-m-l-1)$ from w_1 in the outer cycle of G . Hence, no vertices of G are valid choices for w_3 and w_4 . This gives the desired contradiction that W does not resolve G . Thus, $\beta(G) \geq 5$.

□

The following theorem follows immediately from Theorem 4 and Theorem 5.

Theorem 7 For $n \equiv 1 \pmod{6}$, $\beta(C_n(1, 2, 3)) = 5$.

In [4], Chartrand et al modeled the problem of finding the metric dimension of a graph as an integer programming problem. Using the *Lindo* integer programming software package, we computed the exact metric dimension of the circulant graph $C_n(1, 2, \dots, t)$ for $2 \leq t \leq 5$ and $5 \leq n \leq 22$. The results are summarized in Table 1 and Table 2.

t, n	5	6	7	8	9	10	11	12	13	14
2	4	3	3	3	4	3	3	3	4	3
3	4	5	6	4	4	4	4	4	5	4
4	4	5	6	7	8	5	4	4	5	5
5	4	5	6	7	8	9	10	6	5	6

Table 1: The metric dimension of $C_n(1, 2, \dots, t)$, $5 \leq n \leq 14$.

t, n	15	16	17	18	19	20	21	22
2	3	3	4	3	3	3	4	3
3	4	4	4	4	5	4	4	4
4	6	6	6	5	4	4	5	5
5	6	6	6	6	7	7	8	6

Table 2: The metric dimension of $C_n(1, 2, \dots, t)$, $15 \leq n \leq 22$.

Putting the empirical data in Tables 1 and 2 together with Propositions 1,2 and Theorems 1,4 and 7, we obtain the following theorem.

Theorem 8

(1) For $n \geq 6$ we have

$$\beta(C_n(1, 2)) = \begin{cases} 4 & \text{for } n \equiv 1 \pmod{4} \\ 3 & \text{otherwise} \end{cases}.$$

(2) For $n \geq 8$ we have

$$\beta(C_n(1, 2, 3)) = \begin{cases} 5 & \text{for } n \equiv 1 \pmod{6} \\ 4 & \text{otherwise} \end{cases}.$$

Note that if $n \leq 2t + 1$, then $C_n(1, 2, \dots, t)$ is a complete graph and so the metric dimension is $n - 1$. Theorem 8 might lead one to conjecture that for $n \geq 2t + 2$,

$$\beta(C_n(1, 2, \dots, t)) = \begin{cases} t + 2 & \text{if } n \equiv 1 \pmod{2t} \\ t + 1 & \text{otherwise} \end{cases},$$

but Table 2 shows that this is not the case for $t = 4, 5$. However, the metric dimension of $C_n(1, 2, \dots, t)$ does appear to depend on the congruence class of n modulo $2t$.

3 Metric dimension of Cayley hypergraphs

In this section we bound the metric dimension of Cayley hypergraphs on finite Abelian groups with the canonical set of generators. Specifically, we consider the t -Cayley hypergraph $H = t\text{-Cay}(\Gamma, \Omega)$ where Γ is a finite Abelian group, so we may assume the Γ is a direct product of cyclic groups of prime-power order, say $\Gamma = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_s}$ where n_i is a prime-power for $1 \leq i \leq s$. The canonical set of generators for this group is

$$\Omega = \{(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, 0, \dots, 0, 1)\}$$

and so we require $2 \leq t \leq \max\{n_i : 1 \leq i \leq s\}$. For this Cayley hypergraph H , the 2-section H_2 is isomorphic to the Cartesian product

$$C_{n_1}(1, 2, \dots, t-1) \square C_{n_2}(1, 2, \dots, t-1) \square \cdots \square C_{n_s}(1, 2, \dots, t-1).$$

If $t = 2$, H is a Cayley graph, and we have the following result.

Theorem 9 *Let $H = 2\text{-Cay}(\Gamma, \Omega)$ where $\Gamma = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_s}$ and Ω is the canonical set of generators.*

(1) *If ℓ of the terms in $\{n_1, n_2, \dots, n_s\}$ are odd or equal to 2, then*

$$\beta(H) \leq \begin{cases} s+1, & \text{if } \ell = s \\ 2s - \ell, & \text{if } \ell < s \end{cases}.$$

(2) *If $\Gamma \cong \mathbb{Z}_m \oplus \mathbb{Z}_n$ where $n \geq m \geq 3$, then*

$$\beta(H) = \begin{cases} 3, & \text{if } m \text{ or } n \text{ is odd} \\ 4, & \text{if } m \text{ and } n \text{ are both even} \end{cases}.$$

(3) *If $\Gamma \cong \mathbb{Z}_n \oplus \mathbb{Z}_2$, then*

$$\beta(H) = \begin{cases} 2, & \text{if } n \text{ is odd} \\ 3, & \text{if } n \text{ is even} \end{cases}.$$

Proof:

- (1) The Cayley graph $H \cong C_{n_1} \square C_{n_2} \square \cdots \square C_{n_s}$, and so in this case the result follows from Proposition 3(1) and Proposition 6.
- (2) In this case $H \cong C_m \square C_n$, and so the result follows from Proposition 3(2).

- (3) In this case $H \cong C_n \square K_2$, and so the result follows from Proposition 3(3). □

In the case where $t = \max\{n_i : 1 \leq i \leq s\}$, the 2-section of $H = t\text{-Cay}(\Gamma, \Omega)$ is a Cartesian product of complete graphs, and we obtain the following result.

Theorem 10 *Let $H = t\text{-Cay}(\Gamma, \Omega)$ where $\Gamma = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_s}$, $n_1 \geq n_2 \geq \cdots \geq n_s \geq 2$, and Ω is the canonical set of generators.*

- (1) *If $t = n_1$, then*

$$n_1 - 1 \leq \beta(H) \leq n_1 - 1 + \sum_{i=2}^s \max\{(n_i - 2), 1\}.$$

- (2) *If $s = 2$ and $t = n_1 \geq n_2 \geq 2$ then*

$$\beta(H) = \begin{cases} \lfloor \frac{2}{3}(n_1 + n_2 - 1) \rfloor, & \text{if } n_2 \leq n_1 \leq 2n_2 - 1 \\ n_1 - 1, & \text{if } n_1 \geq 2n_2 - 1 \end{cases}.$$

Proof:

- (1) In this case the 2-section $H_2 \cong K_{n_1} \square K_{n_2} \square \cdots \square K_{n_s}$, and so the result follows from Proposition 7 and Corollary 1.
- (2) The 2-section $H_2 \cong K_{n_1} \square K_{n_2}$, and so the result follows from Propositions 5 and 7. □

We now bound the metric dimension of t -Cayley hypergraphs for $t = 3$ and $t = 4$.

Theorem 11 *Let $t \in \{3, 4\}$ and let $H = t\text{-Cay}(\Gamma, \Omega)$ where $\Gamma = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_s}$, $n_1 \geq n_2 \geq \cdots \geq n_s \geq 2$, and Ω is the canonical set of generators. Then*

$$t + 1 \leq \beta(H) \leq t + 1 + \sum_{i=2}^s \max\{(n_i - 2), 1\}$$

whenever $n_1 \equiv 1 \pmod{2t}$, and

$$t \leq \beta(H) \leq t + \sum_{i=2}^s \max\{(n_i - 2), 1\}$$

otherwise.

Proof: The 2-section is

$$H_2 \cong C_{n_1}(1, 2, \dots, t-1) \square C_{n_2}(1, 2, \dots, t-1) \square \dots \square C_{n_s}(1, 2, \dots, t-1)$$

and so the bounds follow from Propositions 4 and 7 and Theorem 8. \square

We now determine the exact metric dimension of $H = 3\text{-Cay}(\Gamma, \Omega)$ in the case where $\Gamma \cong \mathbb{Z}_2 \oplus \mathbb{Z}_n$ and Ω is the canonical set of generators. The 2-section of this hypergraph is $H_2 \cong C_n(1, 2) \square K_2$. In Theorem 12 we will show that $\beta(C_n(1, 2) \square K_2) = \beta(C_n(1, 2))$, which will yield the exact value for $\beta(H)$ in Theorem 13. First, we examine the structure of the graph $C_n(1, 2, \dots, t) \square K_2$.

Remark 3 *The graph $G = C_n(1, 2, \dots, t) \square K_2$ contains two copies of $C_n(1, 2, \dots, t)$, G_0 and G_1 , where corresponding vertices in G_0 and G_1 are adjacent. The vertices are indexed by an ordered pair (i, j) , where i is either 0 or 1 and $0 \leq j \leq n-1$. The vertices $(0, j)$ make up G_0 , and the vertices $(1, j)$ make up G_1 (Figure 15).*

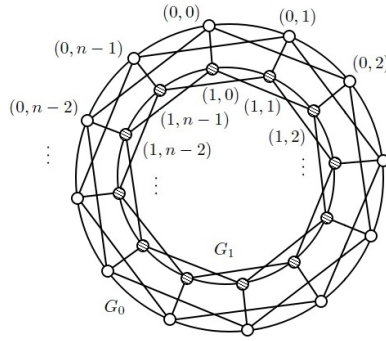


Figure 15: $C_{12}(1, 2) \square K_2$

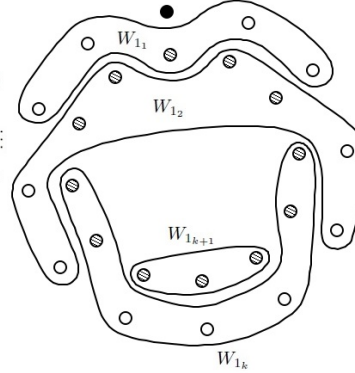


Figure 16: $R_{\{w_1\}}((C_{12}(1, 2) \square K_2))$

Remark 4 *The hyperedges of the resolving hypergraph $R_{\{w_1, w_2\}}(C_n(1, 2) \square K_2)$ cannot be represented as convex shapes (Figure 16). This makes it difficult to tell which vertices share edge neighbourhoods. To make this clear, instead of adding hyperedges W_{2_d} for $1 \leq d \leq k+1$, we add edges between every pair of vertices which are unresolved by both w_1 and w_2 in G (Figure 17).*

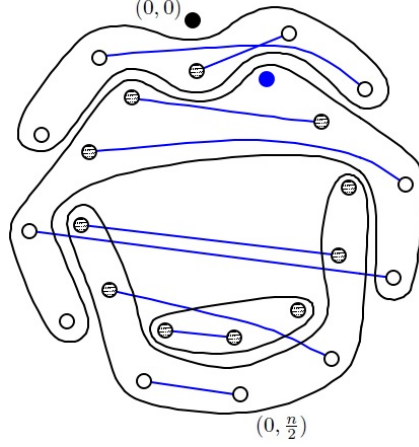


Figure 17: $R_{\{(0,0),(1,1)\}}((C_{12}(1,2)\square K_2)$

Theorem 12 $\beta(C_n(1,2)\square K_2) = \beta(C_n(1,2))$.

Proof: By Proposition 6, $\beta(C_n(1,2)\square K_2) \geq \beta(C_n(1,2))$. It remains to show that $\beta(C_n(1,2)\square K_2) \leq \beta(C_n(1,2))$. Let $G = C_n(1,2)\square K_2$, $n = 4k + r$ for $0 \leq r \leq 3$ and some integer $k \geq 2$. Take $w_1 = (0,0)$ and $w_2 = (1,1)$. The pairs of vertices in G not resolved by either w_1 or w_2 are $\{(1,0), (0,1)\}, \{(0, n-1), (0,2)\}, \{(1, n-1), (1,2)\}, \{(1, n-2), (0,3)\}, \{(0, n-3), (0,4)\}, \{(1, n-3), (1,4)\}, \{(1, n-4), (0,5)\}, \{(0, n-5), (0,6)\}, \{(1, n-5), (1,6)\}, \dots$ (Figure 17).

The unresolved pairs contained in W_{1_k} and $W_{1_{k+1}}$ depend on the value of r . We look at three cases for how this sequence of unresolved pairs ends.

Case 1: $r = 0$. In this case, the final unresolved pairs in G are as follows. $\dots, \{(1, \frac{n}{2} + 2), (0, \frac{n}{2} - 1)\}, \{(0, \frac{n}{2} + 1), (0, \frac{n}{2})\}, \{(1, \frac{n}{2} + 1), (1, \frac{n}{2})\}$ (Figure 17). In this case, taking $w_3 = (0,4)$ resolves all remaining pairs. To show this, let $\{(a,b), (x,y)\}$ be any of the unresolved pairs previously listed. observe that $d((0,4), (a,b)) > d((0,4), (x,y))$ for all of the unresolved pairs, $\{(a,b), (x,y)\}$. Hence, when $r = 0$, $\beta(G) \leq 3 = \beta(C_n(1,2))$.

Case 2: $r = 1$. In this case, the final unresolved pairs in G are as follows. $\dots, \{(1, \frac{n-1}{2} + 3), (0, \frac{n-1}{2} - 1)\}, \{(1, \frac{n-1}{2} + 1), (1, \frac{n-1}{2})\}, \{(1, \frac{n-1}{2} + 2), (1, \frac{n-1}{2})\}, \{(1, \frac{n-1}{2} + 2), (1, \frac{n-1}{2} + 1)\}, \{(0, \frac{n-1}{2} + 1), (0, \frac{n-1}{2})\}, \{(0, \frac{n-1}{2} + 2), (0, \frac{n-1}{2})\}, \{(0, \frac{n-1}{2} + 2), (0, \frac{n-1}{2} + 1)\}$ (Figure 18). As previously, Let $\{(a,b), (x,y)\}$ be any of the unresolved pairs previously listed. Choosing $w_3 = (0,4)$ resolves all remaining unresolved pairs, since $d((0,4), (a,b)) > d((0,4), (x,y))$, except for $\{(1, \frac{n-1}{2} + 2), (1, \frac{n-1}{2} + 1)\}$ and

$\{(0, \frac{n-1}{2} + 2), (0, \frac{n-1}{2} + 1)\}$. Taking $w_4 = (0, 3)$ resolves these last two pairs. Thus, when $r = 1$, $\beta(G) \leq 4 = \beta(C_n(1, 2))$.

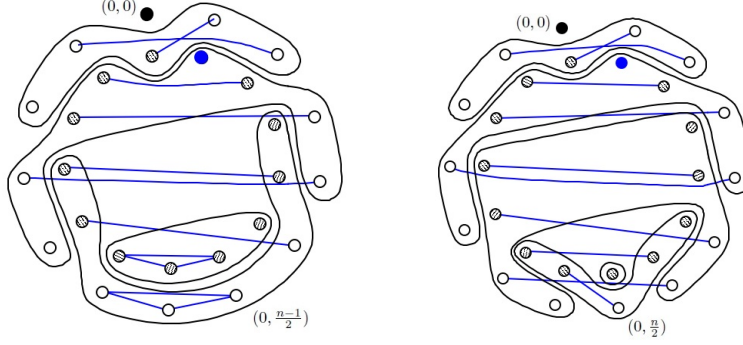


Figure 18: $R_{\{(0,0),(1,1)\}}(C_{13}(1,2) \square K_2)$

Figure 19: $R_{\{(0,0),(1,1)\}}(C_{14}(1,2) \square K_2)$

Case 3: $r = 2$. In this case, the final unresolved pairs in G are as follows. $\dots, \{(1, \frac{n}{2} + 3), (0, \frac{n}{2} - 2)\}, \{(0, \frac{n}{2} + 2), (0, \frac{n}{2} - 1)\}, \{(1, \frac{n}{2} + 2), (1, \frac{n}{2} - 1)\}, \{(1, \frac{n}{2} + 1), (0, \frac{n}{2})\}$ (Figure 19). Let $\{(a, b), (x, y)\}$ be any of the unresolved pairs previously listed. Observe that $d((0, 4), (a, b)) > d((0, 4), (x, y))$ for all of the unresolved pairs, $\{(a, b), (x, y)\}$. Thus, $w_3 = (0, 4)$ resolves all the remaining unresolved pairs in G , and $\beta(G) \leq 3 = \beta(C_n(1, 2))$ whenever $r = 2$.

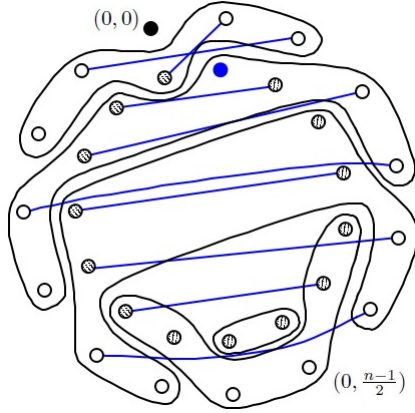


Figure 20: $R_{\{(0,0),(1,1)\}}(C_{15}(1,2) \square K_2)$

Case 4: $r = 3$ In this case, the final unresolved pairs in G are as follows. $\dots, \{(1, \frac{n-1}{2} + 4), (0, \frac{n-1}{2} - 2)\}, \{(0, \frac{n-1}{2} + 3), (0, \frac{n-1}{2} - 1)\}, \{(1, \frac{n-1}{2} + 3), (1, \frac{n-1}{2} - 1)\}$ (Figure 22). As previously, Let $\{(a, b), (x, y)\}$ be any of the unresolved pairs previously listed. Observe that $d((0, 3), (a, b)) > d((0, 3), (x, y))$ for all of the unresolved pairs, $\{(a, b), (x, y)\}$. Thus, $w_3 = (0, 3)$ resolves all the remaining unresolved pairs in G , and $\beta(G) \leq 3 = \beta(C_n(1, 2))$ whenever $r = 3$. Hence, $\beta(C_n(1, 2) \square K_2) \leq \beta(C_n(1, 2))$ for all n .

□

Theorem 13 Let $\Gamma = \mathbb{Z}_2 \oplus \mathbb{Z}_n$, $n \geq 6$, and let $H = 3\text{-Cay}(\Gamma, \Omega)$ where $\Omega = \{(1, 0), (0, 1)\}$. Then

$$\beta(H) = \begin{cases} 4, & n \equiv 1 \pmod{4} \\ 3, & \text{otherwise} \end{cases}$$

Proof: The 2-section $H_2 \cong C_n(1, 2) \square K_2$, and so the result follows from Proposition 7, Theorem 8(1) and Theorem 12.

□

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